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GENERALIZED VECTORS AND THE MATRIX EQUATION $X^{(k)} = B$

by

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Submitted to the Department of Mathematics and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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June, 1952

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Introduction

The set of n-tuples of elements of a field K with addition of n-tuples and multiplication of n-tuples by elements of K defined in the usual manner constitutes an n-dimensional vector space $L_n(K)$ over K, see [7]†. The n-tuples, or vectors, may be represented by $1 \times n$ matrices over K. Then vector addition is just ordinary matrix addition and multiplication of vectors by elements of K becomes multiplication of matrices by elements of K.

If K consists of the ordinary complex numbers (or real numbers), then it is customary to define a bilinear function, called the inner product, of the vectors; and, in terms of the inner product, one can then associate with every vector a norm and with every pair of vectors a distance. In matrix terminology, one may define the inner product of a vector $x = [x_1, x_2, \ldots, x_n]$ with a vector $y = [y_1, y_2, \ldots, y_n]$ to be the determinant of the single-entry matrix resulting upon multiplying the row matrix x on the right by the column matrix y^* which is the complex-conjugate, transposed matrix of y. Furthermore, the components of a vector x are then clearly just the first order minors of its matrix representation.

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† Numbers in brackets refer to the references at the end of the dissertation.

Turning the foregoing discussion around, one sees that the set of $1 \times n$ matrices over a field K forms an n-dimensional vector space over K under ordinary matrix addition and multiplication of matrices by elements of K. The $1 \times n$ matrices can then be regarded as vectors. When K consists of the ordinary complex numbers (or real numbers) the inner product is, as mentioned before, easily defined in terms of matrix multiplication. The norm, or length, of a vector may then be defined to be the square root of the inner product of that vector with itself. From this point of view, it is only reasonable to inquire whether an analogous interpretation may be put upon $k \times n$ matrices, $1 < k \leqq n$, over K. Certainly, $k \times n$ matrices can be added and multiplied by elements of K. Moreover, the analogues of the inner product, norm, and distances are obtained in an obvious manner. It turns out that, whereas the norm of a vector represented by a $1 \times n$ matrix has the dimension of length, the norm of an entity represented by a $2 \times n$ matrix has the dimension of area and, more generally, the norm of the analogue represented by a $k \times n$ matrix is k-dimensional. Therefore it seems reasonable to interpret the $k \times n$ matrices over K as k-dimensional vectors. The components of one of these k-dimensional vectors may be defined to be the k-th order minors of the $k \times n$ matrix representing it. If one arranges the components by the principle under which words are ordered in a dictionary, that is, in lexicographical order, then the row matrix having them as its elements is seen to be the k-th compound of the $k \times n$ matrix representing the k-dimensional vector. But the row matrix of its components is a $1 \times N$ matrix over K, $N = n!/((n-k)!k!)$; and, consequently, represents an ordinary vector in an N dimensional vector space over K. We call this space the associated N-space of the n-dimensional vector space $L_n(K)$ previously defined.

A slightly more sophisticated touch can be lent to the above picture, once a distance has been defined in $L_n(K)$; for then one can introduce a geometry. For simplicity, let K be the field of ordinary complex numbers (or real numbers), and let $L_n(K)$ be referred to the natural basis, or coordinate system, $e_1, e_2, \ldots, e_n$, where e_i is the vector whose i-th component is unity, all others being zero. The geometrical configuration determined by each vector is then a point whose coordinates are just the components of the vector. Each ordered pair of vectors determines a directed line segment with the point determined by the first vector as its initial point and that determined by the second vector as its terminal point. We shall call it a 1-cell. The measures of its projections on the coordinate axes are its components. When a 1-cell is identified only by its components it is indistinguishable from an ordinary vector, i.e., a 1-vector. Indeed we shall call the ordered n-tuple consisting of its components a 1-vector.

Similarly, each ordered triple of vectors determines a triangular shaped, directed, plane magnitude which we shall call a 2-cell. The measures of its projections on the coordinate planes will be called its components. More generally, the convex body determined by an ordered k-tuple of vectors will be called a k-cell; the measures of its projections on the k-dimensional, coordinate hyperplanes will be called its components; and the N-tuple, $N = n!/((n-k)!k!)$, consisting of its components taken in lexicographical order will be called the k-vector determined by the k-cell. Now it turns out that the k-dimensional vector determined by the $k \times n$ matrix whose row vectors are the vectors determining the k-cell has components which are $k!$ times the corresponding components of the k-vector determined by the k-cell. Therefore, the set of all k-vectors is seen to be a subset of the vectors of the associated N-space of $L_n(K)$. When identified by only their components, the k-cells are indistinguishable from ordinary vectors in an N-dimensional, vector space over K. This suggests the manner in which the generalized inner product, norm, and distance functions should be defined for k-cells which, in turn, may be thought of as generalized (i.e., k-dimensional), localized vectors.

For convenience of reference, there are assembled in the first two chapters of the dissertation a number of theorems, which are needed later, concerning matrices and determinants. The better known theorems are merely stated. The more obscure and any new theorems are proved.

In the third chapter, k-cells and k-vectors are introduced, algebraic operations therewith defined, and several representations for the components of a k-cell are found.

The generalized inner product, norm, and distance functions are defined and their properties deduced in the fourth chapter. Also, at least a score of representations for the generalized inner product and more than a dozen representations for the Euclidean volume of a k-cell in a real vector space are developed therein. Some of the formulae are well known; many are new.

It is shown in the fifth chapter that a necessary and sufficient condition that an N-tuple, $N = n!/((n-k)!k!)$, of elements of K be the components of a k-cell is that they satisfy certain quadratic relations which we have named the Grassmann quadratic relations. This is not new, however the proof of the sufficiency of the condition as given therein seems considerably simpler than that given by Hodge and Pedoe [8, pp. 312-315]. The Grassmann quadratic relations are shown to be equivalent to certain symmetric quadratic forms whose matrices are called the special Grassmann matrices. Then the Grassmann matrices are defined to be the set of all matrices which are linear combinations of the special Grassmann matrices. The set of k-vectors is shown to determine a set of quadrics, called the Grassmann quadrics, in the associated N-space and it is further shown that a vector in the associated N-space is a k-vector if, and only if, it annihilates the set of Grassmann matrices, that is, insures the vanishing of the set of quadratic forms whose matrices are the Grassmann matrices. Finally certain duality principles are established for Grassmann matrices.

Every linear transformation on $L_n(K)$ induces on the associated N-space an associated linear transformation whose matrix is the k-th compound of that of the original transformation. In the sixth chapter, the effects on the components of k-cells and on the set of k-vectors of various linear transformations on $L_n(K)$, as well as those on the associated N-space, are investigated. Several interesting and unexpected results are obtained. For example, it is shown that whenever $n = 2k$ there are linear transformations on the associated N-space which transform the set of k-vectors into itself and whose matrices are not k-th compounds. Since all linear transformations on the associated N-space whose matrices are k-th compounds of $n \times n$ matrices over K transform the set of k-vectors into itself, this shows that whenever $n = 2k$ the group, which we shall call the k-th compound group, consisting of all linear transformations on the associated N-space whose matrices are k-th compounds of non-singular $n \times n$ matrices is a <u>proper</u> subgroup of that, which we shall call the Grassmann group, consisting of all non-singular linear transformations on the associated N-space under which the intersection of the

Grassmann quadrics remains invariant, in other words, those linear transformations which carry the set of k-vectors into itself. This naturally raises the question of when is a matrix a k-th compound. This question is completely answered, except for the singular case in which the matrix under consideration has rank k, in the seventh and last chapter. Now, as might be expected, one of the conditions of the set which constitutes a complete set of necessary and sufficient conditions that a matrix be a k-th compound is that its row vectors (and also its column vectors) be k-vectors, see (7.4.1.1). As mentioned before, the necessary and sufficient condition for this is that their components satisfy the Grassmann quadratic relations, a proof of which is to be found in Hodge and Pedoe [8, pp. 310-315]. However, the proof of the necessity of this condition as given on pages 246 and 247 of the dissertation seems to be simpler than that in Hodge and Pedoe's book. Actually they have solved only the simplest case; namely, they have found the necessary and sufficient conditions that a $1 \times N$ matrix, $N = n!/((n-k)!k!)$, be the k-th compound of a $k \times n$ matrix. Whereas, in the seventh chapter of the dissertation, the necessary and sufficient conditions that an $M \times N$ matrix, $M = m!/((m-k)!k!)$ and $N = n!/((n-k)!k!)$, be the k-th compound of an $m \times n$ matrix are determined. This is a problem not heretofore solved.

Finally, although many acknowledgments ought to be

made, at least two cannot be omitted. Firstly, the dissertation would never have been completed had it not been for the many invaluable suggestions and the continued encouragement offered by Professor G. B. Price, at whose suggestion it was undertaken; secondly, it is entirely due to the patience, interest, and persistence of my wife, Dorothy, who spent countless hours at the typewriter, that it was ever typed.

Chapter 1

Compound Matrices: Dual Theorems

Let K be a commutative field without characteristic. An $m \times n$ matrix over K is an array

$$A = \begin{bmatrix} a^1_1 & a^1_2 & \cdot & a^1_n \\ a^2_1 & a^2_2 & \cdot & a^2_n \\ \cdot & \cdot & \cdot & \cdot \\ a^m_1 & a^m_2 & \cdot & a^m_n \end{bmatrix} = \begin{bmatrix} a^i_j \end{bmatrix}, \qquad \begin{matrix} i = 1,2,\ldots,m, \\ j = 1,2,\ldots,n, \end{matrix}$$

of mn elements of K, arranged in rectangular form. We assume the elementary theory of matrices and their common properties without further ado. However, it is expedient to introduce some notation.

1.1. Notation. Except when the contrary is explicitly stated, we employ the following symbols.

(1.1.1) $I_n = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 \end{bmatrix} = \begin{bmatrix} \delta^i_j \end{bmatrix}$, where δ^i_j is the Kronecker delta, will denote the $n \times n$ unit matrix over K.

(1.1.2) 0_n will sometimes be used to denote the $n \times n$ matrix with zero entries.

(1.1.3) A' will denote the "transpose of A".

(1.1.4) When K is the field of complex numbers, $\bar{A}$ will denote the matrix whose elements are the complex conjugates of the corresponding elements of A and A^* will denote $\bar{A}'$. When A is real, A^* will denote A'.

(1.1.5) If A is square, then $|A|$ will denote the determinant of A.

(1.1.6) The inverse of A, when it exists, will be denoted by A^{-1}.

(1.1.7) The symbol $C(n,k)$ will mean "the number of combinations of n things taken k at a time". Hence,

(1.1.7.1) $\quad C(n,k) = n(n-1)\ldots(n-k+1)/k!\ , \quad 0 < k \leq n.$

For convenience, we set

(1.1.7.2) $\quad C(n,0) = 1\ , \ 0 \leq n \quad , \ n$ an integer,

(1.1.7.3) $\quad C(n,k) = 0\ , \ 0 \leq n < k\ , \ n$ and k integers.

<u>1.2. Compound Matrices.</u> Let A be an $m \times n$ matrix over K. Let a matrix be formed the elements of which are minors of A of order k, $0 < k \leq \min(m,n)$. Let all minors which come from the same set of k rows (or columns) of A be placed in the same row (or column), and let the priority of elements in rows or columns be decided on the principle by which words are ordered in a dictionary; that is, let the order be "lexicographical" order. The resulting matrix

is called the k-th compound of A, and it is denoted by $A^{(k)}$. It is easy to show that $A^{(k)}$ is a $C(m,k)xC(n,k)$ matrix over K. For completeness, we set

$$(1.2.1) \qquad A^{(0)} = I_1 ,$$

$$(1.2.2) \qquad A^{(k)} = 0_1 , \quad \max(m,n) < k.$$

If $m < k \leq n$ (or $n < k \leq m$), we take $A^{(k)}$ to be the $1xC(n,k)$ (or the $C(n,k) \times 1$) null matrix. We note in passing that $A^{(1)} = A$ and, if A is nxn, $A^{(n)} = [|A|]$.

<u>1.3. Adjoint Compound Matrices.</u> When A is square of order nxn, every minor of order k in $|A|$ is accompanied, in a Laplacian expansion of $|A|$, by its cofactor or signed minor of order $n-k$. If every element of $A^{(k)}$ be replaced by its cofactor in $|A|$ and the resulting matrix be transposed, a matrix of the same order as $A^{(k)}$ is obtained which will be called the k-th adjoint compound of A and will be denoted by $\operatorname{adj}^{(k)}A$. For completeness, we set

$$(1.3.1) \qquad \operatorname{adj}^{(0)}A = [|A|] , \qquad \operatorname{adj}^{(n)}A = I_1 .$$

We usually write $\operatorname{adj} A$ instead of $\operatorname{adj}^{(1)}A$.

(1.3.2) <u>Remark.</u> The elements of $\operatorname{adj}^{(k)}A$ are signed minors of $|A|$ of order $n-k$ arranged in reversed lexicographical order.

<u>1.4. Preliminary Theorems.</u> We list here, without proof, several properties of compound and adjoint compound matrices. For proofs, see [1, pp. 90-110], unless otherwise indicated.

(1.4.1) $$I_n^{(k)} = I_{C(n,k)}.$$

(1.4.2) $$(A^*)^{(k)} = (A^{(k)})^*.$$

(1.4.3) $$(A^{-1})^{(k)} = (A^{(k)})^{-1}.$$

(1.4.4) If the rank of A is r, then the rank of $A^{(k)}$ is $C(r,k)$, see [5].

(1.4.5) $$(AB)^{(k)} = A^{(k)}\, B^{(k)}.$$

(1.4.6) $$\mathrm{adj}^{(k)}(AB) = \mathrm{adj}^{(k)}B\ \mathrm{adj}^{(k)}A.$$

If A is of order $n\mathrm{x}n$, then the following relations are true.

(1.4.7) $$A^{(k)}\mathrm{adj}^{(k)}A = \mathrm{adj}^{(k)}A\ \ A^{(k)} = |A| \cdot I_{C(n,k)}.$$

(1.4.8) $$A^{(k)}(\mathrm{adj}\ A)^{(k)} = (\mathrm{adj}\ A)^{(k)}A^{(k)} = |A|^k I_{C(n,k)}.$$

(1.4.9) $$|A^{(k)}| = |A|^{C(n-1,k-1)}.$$

(1.4.10) $$|\mathrm{adj}^{(k)}A| = |A|^{C(n-1,k)}.$$

We shall need a theorem due to John Williamson, see [12].

(1.4.11) <u>Theorem.</u> If A is an $m \times n$ matrix of rank r, the necessary and sufficient condition that $A^{(k)} = B^{(k)}$ is that

(1) rank of $B < k$ when $r < k$;

(2) there exist non-singular matrices C and D such that

$$CAD = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}, \quad CBD = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix},$$

where T and S are non-singular matrices of order r such that $|T| = |S|$, when $r = k$;

(3) $A = \omega B$, where ω is a k-th root of unity, when $r > k$.

If A is an $n \times n$ matrix, then

(1.4.12) $$\operatorname{adj} A^{(k)} = |A|^{C(n-1,k-1)-1} \operatorname{adj}^{(k)} A,$$

provided we take $|A|^0 = 1$ in all cases.

<u>Proof:</u> We consider the various possibilities separately as cases and subcases.

<u>Case 1.</u> $|A| \neq 0$. Here we have, by (1.4.7),

$$\begin{aligned} |A|\, A^{(k)} \operatorname{adj} A^{(k)} &= |A|\, |A^{(k)}|\, I_{C(n,k)} \\ &= |A^{(k)}|\, A^{(k)} \operatorname{adj}^{(k)} A \end{aligned}$$

Multiplying through by $|A|^{-1}(A^{(k)})^{-1}$ and using (1.4.9) leads to the desired result.

<u>Case 2.</u> $|A| = 0$. We consider several subcases.

<u>Case 2.1</u> $k = 0$. Then, by (1.2.1), $A^{(k)} = I_1$, and so, by (1.3.1), adj $A^{(k)} = I_1$. Therefore, the left member of (1.4.12) is equal to I_1. By (1.3.1), $\text{adj}^{(k)}A = [|A|] = |A|I_1$. While $|A|^{C(n-1,k-1)-1} = |A|^{C(n-1,-1)-1}$. Recalling that $C(n,k) = C(n,n-k)$, we have

$$\begin{aligned} C(n-1,-1) &= C(n-1,n) \\ &= 0 \qquad \text{by } (1.1.7.3). \end{aligned}$$

Hence $|A|^{C(n-1,k-1)-1} = |A|^{0-1} = |A|^{-1}$.

Therefore the right member is equal to $|A|^{-1}|A|I_1 = I_1$.

Hence (1.4.12) holds in this case.

<u>Case 2.2</u> $k = 1$. In this case (1.4.12) becomes

$$\begin{aligned} \text{adj } A &= |A|^{C(n-1,0)-1}\text{adj } A \\ &= |A|^0 \text{ adj } A \qquad \text{by } (1.1.7.2) \\ &= \text{adj } A \qquad \text{by hypothesis,} \end{aligned}$$

which is an obvious identity.

<u>Case 2.3</u> $k = n$. In this case (1.4.12) becomes

$$\begin{aligned} \text{adj } A^{(n)} &= |A|^{C(n-1,n-1)-1} \text{ adj}^{(n)}A \\ &= |A|^0 \text{ adj}^{(n)}A \\ &= \text{adj}^{(n)}A \qquad \text{by hypothesis} \\ &= I_1 \qquad \text{by } (1.3.1). \end{aligned}$$

Hence (1.4.12) holds if

$$\text{adj } [|A|] = I_1 \quad , \quad \text{since } A^{(n)} = [|A|] .$$

Now this is true by 1.3.1. Thus (1.4.12) is true in this case also.

Case 2.4 $1 < k < n$. In this case (1.4.12) becomes $\operatorname{adj} A^{(k)} = 0 \operatorname{adj}^{(k)} A$ since, for $1 < k < n$, $C(n-1,k-1)-1 > 0$, thereby insuring that $|A|^{C(n-1,k-1)-1} = 0$ (recall that $|A| = 0$). Hence we need merely prove that $\operatorname{adj} A^{(k)} = 0_{C(n,k)}$ when $|A| = 0$ and $1 < k < n$. Since $|A| = 0$, we know that $r < n$, where r is the rank of A. By (1.4.4), then, the rank of $A^{(k)}$ is $C(r,k)$. Now, since $r < n$, $C(r,k) < C(n,k)-1$. This means that $\operatorname{adj} A^{(k)}$ has rank zero; in other words,

$$\operatorname{adj} A^{(k)} = 0_{C(n,k)}.$$

This completes the proof in all cases.

1.5. The i,jth Position of an Element of $A^{(k)}$.

Let

$$A = \begin{bmatrix} a_1^1 & a_2^1 & \cdot & a_n^1 \\ a_1^2 & a_2^2 & \cdot & a_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ a_1^m & a_2^m & \cdot & a_n^m \end{bmatrix}$$

be an $m \times n$ matrix over K. If

$$A_j^i = \begin{vmatrix} a_{j_1}^{i_1} & a_{j_2}^{i_1} & \cdot & a_{j_k}^{i_1} \\ a_{j_1}^{i_2} & a_{j_2}^{i_2} & \cdot & a_{j_k}^{i_2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{j_1}^{i_k} & a_{j_2}^{i_k} & \cdot & a_{j_k}^{i_k} \end{vmatrix} ,$$

for the prescribed set of indices

$$1 \leqq i_1 < i_2 < \dots < i_k \leqq m, \quad 1 \leqq j_1 < j_2 < \dots < j_k \leqq n,$$

in the $C(m,k) \times C(n,k)$ matrix

$$A^{(k)} = \begin{bmatrix} A_1^1 & \cdot & A_{C(n,k)}^1 \\ \cdot & \cdot & \cdot \\ A_1^{C(m,k)} & \cdot & A_{C(n,k)}^{C(m,k)} \end{bmatrix} ,$$

then

$$i = C(m,k) - \sum_{\nu=1}^{k} C(m-i_\nu, k-\nu+1) ,$$

(1.5.1)

$$j = C(n,k) - \sum_{\nu=1}^{k} C(n-j_\nu, k-\nu+1) .$$

Proof: The same argument applies for both parts of the conclusion. Hence we need only prove the statement for j. In view of 1.2, the problem is equivalent to that of determining the number j of k-tuples $(b_1, b_2, \dots, b_k)$ of natural numbers such that $b_1 < b_2 < \dots < b_k$, up to and including the k-tuple $(j_1, j_2, \dots, j_k)$ in the $C(n,k)$-term sequence

$$(1.5.2) \qquad (1,\ 2,\ \ldots,\ k),\ (1,\ 2,\ \ldots,\ k-1,\ k+1),\ \ldots,$$
$$(j_1,\ j_2,\ \ldots,\ j_k),\ \ldots,\ (n-k+1,\ \ldots,\ n)$$

of all such k-tuples that can be formed from the first n natural numbers, where the order is lexicographical order. First, we find the number of terms of (1.5.2) between the arbitrary pair

$$(1.5.3) \qquad \begin{aligned} &(j_1,\ j_2,\ \ldots,\ j_{v-1},\ j_{v-1}+1,\ \ldots,\ j_{v-1}+k-v+1), \\ &(j_1,\ j_2,\ \ldots,\ j_{v-1},\ j_v,\ j_v+1,\ \ldots,\ j_v+k-v). \end{aligned}$$

Let t_v be any natural number such that $j_{v-1} < t_v < j_v$ and consider the k-tuples, in (1.5.2), of the form

$$(1.5.4) \qquad (j_1,\ \ldots,\ j_{v-1},\ t_v,\ b_{v+1},\ \ldots,\ b_k).$$

Since $t_v < b_{v+1} < b_{v+2} < \ldots < b_k$, we are constrained to select the numbers $b_{v+1},\ \ldots,\ b_k$ from the $n-t_v$ numbers t_v+1, t_v+2, ..., n. Any combination of $k-v$ numbers of this set, arranged in the order of increasing magnitude, will serve to fill the last $k-v$ places in (1.5.4). There are exactly $C(n-t_v,k-v)$ such combinations. Hence, for each number t_v, such that $j_{v-1} < t_v < j_v$, there are $C(n-t_v,k-v)$ terms of the form (1.5.4) in the interval (1.5.3). Summing these we conclude that there are

$$(1.5.5) \qquad \sum_{t_v=j_{v-1}+1}^{j_v-1} C(n-t_v,k-v)$$

terms of (1.5.2) in the interval (1.5.3) if we count the first but not the last term. Now this is true for

$v = 1, 2, \ldots, k$ if we set $j_0 = 0$. Summing this for all values of v, we see that there are

$$\sum_{v=1}^{k} \sum_{t_v=j_{v-1}+1}^{j_v-1} C(n-t_v, k-v) \tag{1.5.6}$$

terms of (1.5.2) preceding the term $(j_1, j_2, \ldots, j_k)$. Hence it follows that

$$j = 1 + \sum_{v=1}^{k} \sum_{t_v=j_{v-1}+1}^{j_v-1} C(n-t_v, k-v)$$

$$= 1 + \sum_{v=1}^{k} \sum_{u=n-j_v+1}^{n-j_{v-1}-1} C(u, k-v)$$

$$= 1 + \sum_{v=1}^{k} \left(\sum_{u=k-v}^{n-j_{v-1}-1} C(u, k-v) - \sum_{u=k-v}^{n-j_v} C(u, k-v) \right)$$

$$= 1 + \sum_{v=1}^{k} \Big(C(n-j_{v-1}, k-v+1) - C(n-j_v+1, k-v+1) \Big)$$

$$= 1 + C(n-j_0, k) + \sum_{v=1}^{k} \Big(C(n-j_v, k-v) - C(n-j_v+1, k-v+1) \Big) - C(n-j_k+1, 1)$$

$$= 1 + C(n, k) - \left(\sum_{v=1}^{k-1} C(n-j_v, k-v+1) \right) - \Big(C(n-j_k, 1) + C(n-j_k, 0) \Big)$$

$$= C(n, k) - \sum_{v=1}^{k} C(n-j_v, k-v+1).$$

This completes the proof in which we made use of the two standard formulae $C(n,r)+C(n,r+1) = C(n+1,r+1)$ and $\sum_{u=r}^{n} C(u,r) = C(n+1,r+1)$; we also made use of the fact that, by hypothesis, $j_0 = 0$.

<u>1.6. Duality in Lexicographical Ordering.</u> Consider the $C(n,k)$-term sequence

$$(1.6.1) \qquad (1,\ldots,k),\ldots,(n-k+1,\ldots,n)$$

of all k-tuples $(b_1,\ldots,b_k)$ of the first n natural numbers such that $b_1 < \ldots < b_k$, where the order of terms in (1.6.1) is lexicographical order, and

$$(1.6.2) \qquad (1,\ldots,n-k),\ldots,(k+1,\ldots,n)$$

of all (n-k)-tuples $(c_1,\ldots,c_{n-k})$ of the first n natural numbers such that $c_1 < \ldots < c_{n-k}$, arranged in lexicographical order. To each term $(b_1,\ldots,b_k)$ in (1.6.1) there corresponds a unique (n-k)-tuple $(b'_1,\ldots,b'_{n-k})$ such that $b'_1 < \ldots < b'_{n-k}$, namely, the (n-k)-tuple consisting of the elements, taken in order of increasing magnitude, of the complementary set $\{b'_i\}$ in the set of the first n natural numbers of the set $\{b_i\}$ of natural numbers which make up the k-tuple $(b_1,\ldots,b_k)$; and conversely. Hence if $(i_1,\ldots,i_k)$ runs through (1.6.1) from left to right then $(i'_1,\ldots,i'_{n-k})$ runs through (1.6.2), in fact, $(i'_1,\ldots,i'_{n-k})$ runs through (1.6.2) from right to left, as the following argument shows. If $(i_1,\ldots,i_k)$ is any term of (1.6.1) then $(i'_1,\ldots,i'_{n-k})$ is clearly a term of (1.6.2). If $(i_1,\ldots,i_k)$ and $(j_1,\ldots,j_k)$ are any two distinct terms of (1.6.2) then $(i'_1,\ldots,i'_{n-k})$ and $(j'_1,\ldots,j'_{n-k})$ are likewise distinct terms of (1.6.2), because

$(i_1,\dots,i_k) \neq (j_1,\dots,j_k)$ implies that, for some u, $1 \leqq u \leqq k$, $i_u \neq k_v$ for $v = 1,\dots,k$, which, in turn, implies that for some w, $1 \leqq w \leqq n-k$, $j'_w = i_u$, but this implies that $(i'_1,\dots,i'_{n-k}) \neq (j'_1,\dots,j'_{n-k})$ since, clearly, $i'_w \neq i_u$. It follows, therefore, that $(i'_1,\dots,i'_{n-k})$ takes on the value of every term of (1.6.2) if $(i_1,\dots,i_k)$ takes on the value of every term of (1.6.1). Now when $(i_1,\dots,i_k) = (1,\dots,k)$ we have $(i'_1,\dots,i'_{n-k}) = (k+1,\dots,n)$; that is, when $(i_1,\dots,i_k)$ is the first term of (1.6.1), $(i'_1,\dots,i'_{n-k})$ is the last term of (1.6.2). We complete the argument by showing that if, for any pair $(i_1,\dots,i_k)$, $(j_1,\dots,j_k)$, the term $(j_1,\dots,j_k)$ appears to the right of the term $(i_1,\dots,i_k)$ in (1.6.1) then the term $(j'_1,\dots,j'_{n-k})$ appears to the left of the term $(i'_1,\dots,i'_{n-k})$ in (1.6.2). To this end, suppose that $(j_1,\dots,j_k)$ does appear to the right of $(i_1,\dots,i_k)$ in (1.6.1). Then, because the terms of (1.6.1) are arranged in lexicographical order, there must be a natural number u such that both

(1.6.3) $$i_u < j_u, \qquad 1 \leqq u \leqq k,$$

(1.6.4) $$i_v = j_v, \qquad v = 1,2,\dots,u-1,$$

hold. Now, since $i_1 < \dots < j_k$ and $j_1 < \dots < j_k$, we have, in view of (1.6.3) and (1.6.4),

(1.6.5) $$j_1 < \dots < j_{u-1} < i_u < j_u < \dots < j_k.$$

Now it is clear, from (1.6.5), that i_u must be one of the natural numbers $j'_1,\dots,j'_{n-k}$, say

$$(1.6.6) \qquad i_u = j'_w, \qquad 1 \leqq w \leqq n-k.$$

It follows, therefore, that the set $j'_1,\dots,j'_{w-1}$ consists of all natural numbers less than i_u and not in the set $j_1,\dots,j_k$. More specifically, since $j_1 < \dots < j_k$, we have, in view of (1.6.3), $j'_1,\dots,j'_{w-1}$ is the set of all natural numbers less than i_u and not in the set $j_1,\dots,j_{u-1}$. Now, taking into account (1.6.4), we have $j'_1,\dots,j'_{w-1}$ is the set of all natural numbers less than i_u and not in the set $i_1,\dots,i_{u-1}$. Hence the set $j'_1,\dots,j'_{w-1}$ must be also the set $i'_1,\dots,i'_{w-1}$. Now, remembering that $i'_1 < \dots < i'_{n-k}$ and $j'_1 < \dots < j'_{n-k}$, we are forced to the conclusion that both

$$(1.6.7) \qquad i'_v = j'_v, \qquad v = 1,2,\dots,w-1, \text{ and}$$

$$(1.6.8) \qquad i'_w \geqq i_u.$$

But we know that $i'_w \neq i_u$, hence we have, from (1.6.8),

$$i'_w > i_u.$$

Then, by (1.6.6), we have

$$(1.6.9) \qquad i'_w > j'_w .$$

The relations (1.6.7) and (1.6.9) show that, since the terms of (1.6.2) are arranged in lexicographical order, the term $(j_1',\dots,j_{n-k}')$ appears to the left of the term $(i_1',\dots,i_{n-k}')$ in (1.6.2). We have proved, therefore, that if $(i_1,\dots,i_k)$ is the i-th term, counting from the left, in the sequence (1.6.1) then $(i_1',\dots,i_{n-k}')$ is the i-th term, counting from the right, in the sequence (1.6.2). It follows that if $(i_1',\dots,i_{n-k}')$ is the i'-th term, counting from the left, of the sequence (1.6.2) then

$$i' = C(n,k) + 1 - i. \tag{1.6.10}$$

The following corollary is an immediate consequence of (1.6.10) and (1.5.1).

(1.6.11) <u>Corollary.</u> If $i_1 < \dots < i_k$, $i_1' < \dots < i_{n-k}'$, and the set $\{i_1,\dots,i_k,i_1',\dots,i_{n-k}'\}$ is some permutation of the first n natural numbers, then

$$\sum_{\nu=1}^{k} C(n-i_\nu,k-\nu+1) + \sum_{\nu=1}^{n-k} C(n-i_\nu',n-k-\nu+1) = C(n,k) - 1.$$

1.7. Definition of the Matrix $G_{n,k}$.

Under the correspondence

$$\text{(1.7.1)}\quad \begin{aligned} i &\leftrightarrow (i_1,\dots,i_k), & i &= C(n,k) - \sum_{\nu=1}^{k} C(n-i_\nu,k-\nu+1), \\ j &\leftrightarrow (j_1,\dots,j_k), & j &= C(n,k) - \sum_{\nu=1}^{k} C(n-j_\nu,k-\nu+1), \end{aligned}$$

we have seen, (1.5.1), that when $(i_1,\dots,i_k)$ and $(j_1,\dots,j_k)$ run through the terms of the sequence (1.6.1), from left to right, i and j run through the sequence

$1,2,\ldots,C(n,k)$, from left to right. Now let

$$(1.7.2)\qquad p_i = \sum_{\nu=1}^{k} i_\nu\ , \quad p_j = \sum_{\nu=1}^{k} j_\nu\ , \quad q_k = \tfrac{1}{2}k(2n-k+1),$$

$$i,j = 1,2,\ldots,C(n,k)\ , \qquad 0<k \leqq n.$$

Finally, let $G_{n,k}$ be a $C(n,k)\mathrm{x}C(n,k)$ matrix with all elements zero except those in the secondary diagonal, these elements being $(-1)^{p_j+q_k}$ in the $C(n,k)+1-j$ column, $j = 1,2,\ldots,C(n,k)$. In other words, we have, for $0<k \leqq n$,

$$G_{n,k} = \left[(-)^{p_j+q_k}\delta^{i}_{C(n,k)+1-j}\right] \qquad i,j = 1,\ldots,C(n,k).$$

For completeness, we define

$$(1.7.3)\qquad i = j = 1, \quad p_i = p_j = 0, \quad q_k = 0 \text{ when } k = 0.$$

Then, recalling (1.1.7.2), we have

$$(1.7.4)\qquad G_{n,0} = \left[(-)^{0+0}\delta^{1}_{1+1-1}\right] = I_1.$$

This completes the definition of $G_{n,k}$, which we restate here.

(1.7.5) <u>Definition.</u> $G_{n,k} = \left[(-)^{p_j+q_k}\delta^{i}_{C(n,k)+1-j}\right]$,

$i,j = 1,\ldots,C(n,k)$, $0 \leqq k \leqq n$, where p_j, q_k are defined in (1.7.2) and (1.7.3).

<u>1.8. Properties of $G_{n,k}$.</u> We have, using (1.7.2) in (1.7.5),

$$G_{n,n} = \left[(-)^{\frac{1}{2}n(n+1)+\frac{1}{2}n(n+1)}\delta^{1}_{1+1-1}\right] = I_1.$$

Hence, recalling (1.7.4), we see that

(1.8.1) $\quad G_{n,n} = G_{n,0} = I_1$ for all $n \geqq 0$.

Since the row vectors, as well as the column vectors, of $G_{n,k}$ have Euclidean length 1, it follows that $G_{n,k}$ is an orthogonal matrix; in other words,

(1.8.2) $\quad G_{n,k}G'_{n,k} = I_{C(n,k)}$ for all $n \geqq 0$.

In order to obtain the transposed matrix of $G_{n,k}$, we replace its rows by its columns and vice-versa. Hence

(1.8.3) $\quad$ for $i,j = 1,\ldots,C(n,k)$, when $0 \leqq k \leqq n$,

$$G'_{n,k} = \left[(-)^{p_i+q_k}\delta_j^{C(n,k)+1-i}\right] .$$

Let $(r_1,\ldots,r_{n-k})$ and $(s_1,\ldots,s_{n-k})$ be respectively the r-th and the s-th terms of (1.6.2) and let $(r'_1,\ldots,r'_k)$ and $(s'_1,\ldots,s'_k)$ be their respective complements in (1.6.1). Using (1.7.5) and recalling that $C(n,n-k) = C(n,k)$, we have

$$G_{n,n-k} = \left[(-)^{p_s+q_{n-k}}\delta^r_{C(n,k)+1-s}\right]$$

$$= (-1)^{q_{n-k}}\left[(-1)^{s_1+\cdots+s_{n-k}}\delta^r_{C(n,k)+1-s}\right].$$

Now

$$(-1)^{s_1+\cdots+s_{n-k}} = (-1)^{s_1+\cdots+s_{n-k}+s'_1+\cdots+s'_k+s'_1+\cdots+s'_1} .$$

But the set $s_1, \ldots, s_{n-k}, s_1' + \ldots + s_k'$ is some permutation of the first n natural numbers; so

$$(-1)^{s_1 + \ldots + s_{n-k}} = (-1)^{\frac{1}{2}n(n+1) + (s_1' + \ldots + s_k')}$$

Therefore

$$G_{n,n-k} = (-1)^{q_{n-k}} (-1)^{\frac{1}{2}n(n+1) + (s_1' + \ldots + s_k')} \delta^r_{C(n,k)+1-s}$$

$$= (-1)^{\frac{1}{2}n(n+1) + q_{n-k}} (-1)^{(s_1' + \ldots + s_k')} \delta^r_{C(n,k)+1-s} .$$

Now, by the argument of 1.6, we know that, as r, s run through the sequence $1, \ldots, C(n,k)$, $(s_1', \ldots, s_k')$ runs through the sequence (1.6.1) from right to left. Hence if we replace r, s respectively by i, j, where $i \leftrightarrow (i_1, \ldots, i_k)$ and $j \leftrightarrow (j_1, \ldots, j_k)$, then, by (1.6.10), we must replace $(s_1' + \ldots + s_k')$ by $p_{C(n,k)+1-j}$, in agreement with (1.7.2). Hence we may write

$$G_{n,n-k} = (-1)^{\frac{1}{2}n(n+1) + q_{n-k}} (-1)^{p_{C(n,k)+1-j}} \delta^i_{C(n,k)+1-j} .$$

Since $\delta^i_{C(n,k)+1-j} = 0$ except when $C(n,k)+1-j = i$,

we have

$$(-1)^{p_{C(n,k)+1-j}} \delta^i_{C(n,k)+1-j} = (-1)^{p_i} \delta^i_{C(n,k)+1-j} .$$

Using this, we have

$$G_{n,n-k} = (-1)^{\frac{1}{2}n(n+1)+q_{n-k}}\left[(-1)^{p_i}\delta^{i}_{C(n,k)+1-j}\right]$$

$$= (-1)^{\frac{1}{2}n(n+1)+q_{n-k}+q_k}\left[(-1)^{p_i+q_k}\delta^{i}_{C(n,k)+1-j}\right].$$

Now we note that $\delta^{i}_{C(n,k)+1-j} = \delta^{C(n,k)+1-i}_{j}$ and

$$(-1)^{\frac{1}{2}n(n+1)+q_{n-k}+q_k}$$

$$= (-1)^{\frac{1}{2}n(n+1)+\frac{1}{2}(n-k)(n+k+1)+\frac{1}{2}k(2n-k+1)}$$

$$= (-1)^{k(n-k)} .$$

Hence

$$G_{n,n-k} = (-1)^{k(n-k)}\left[(-)^{p_i+q_k}\delta^{C(n,k)+1-i}_{j}\right] .$$

Thus, in view of (1.8.3), we have proved

(1.8.4) $\qquad G_{n,n-k} = (-)^{k(n-k)} G'_{n,k}, \qquad 0 \leqq k \leqq n.$

Taking $n = 2k$, we have, as a corollary to (1.8.4),

(1.8.5) $\qquad G_{2k,k} = (-)^{k} G'_{2k,k} \quad$ for all $\quad k \geqq 0.$

This shows us that $G_{2k,k}$ is symmetric or skew-symmetric according as k is an even or an odd natural number. Now we state, without proof, another property.

$$(1.8.6) \qquad G_{n,k} = \begin{bmatrix} 0 & & & & G_{n-1,k-1} \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & (-)^{k(i-1)} G_{n-i,k-1} & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ (-)^{k(n-k)} G_{k-1,k-1} & & & & 0 \end{bmatrix}.$$

From (1.8.6), we have

$$(-)^k G_{n-1,k} = \begin{bmatrix} 0 & & & & (-)^k G_{n-2,n-1} \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & (-)^{k(i-1)} G_{n-i,k-1} & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ (-)^{k(n-k)} G_{k-1,k-1} & & & & 0 \end{bmatrix}.$$

Substituting the foregoing in

$$G_{n,k} = \begin{bmatrix} 0 & & & & & G_{n-1,k-1} \\ & & & & (-)^k G_{n-2,k-1} & \\ & & & \cdot^{\cdot^{\cdot}} & & \\ & & (-)^{k(i-1)} G_{n-i,k-1} & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ (-)^{k(n-k)} G_{k-1,k-1} & & & & & 0 \end{bmatrix}$$

yields the following corollary to (1.8.6).

$$(1.8.7) \qquad G_{n,k} = \begin{bmatrix} 0 & G_{n-1,k-1} \\ (-)^k G_{n-1,k} & 0 \end{bmatrix} \qquad 0 < k \leqq n.$$

<u>1.9. Some Duality Relations.</u> Let the matrix $A = \left[a_j^i \right]$, $i,j = 1,2,\ldots,n$, be an $n \times n$ matrix over K.

Then, by 1.2, $A^{(k)} = \left[A^i_j\right]$, where the elements

$$A^i_j = \begin{vmatrix} a^{i_1}_{j_1} & \cdot & a^{i_1}_{j_k} \\ \cdot & \cdot & \cdot \\ a^{i_k}_{j_1} & \cdot & a^{i_k}_{j_k} \end{vmatrix}$$

are $k\times k$ minors of A, arranged in lexicographical order, with

$$1 \leqq i_1 < \cdots < i_k \leqq n, \qquad 1 \leqq j_1 < \cdots < j_k \leqq n$$

and

$$i = C(n,k) - \sum_{\nu=1}^{k} C(n-i_\nu, k-\nu+1), \qquad j = C(n,k) - \sum_{\nu=1}^{k} C(n-j_\nu, k-\nu+1)$$

in accordance with 1.5. Now, using (1.7.5) and (1.8.3), we have

$$G'_{n,k}A^{(k)}G_{n,k} = \left[(-)^{p_i+q_k}\delta_j^{C(n,k)+1-i}\right]\left[A^i_j\right]\left[(-)^{p_j+q_k}\delta^i_{C(n,k)+1-j}\right]$$

$$= \left[(-)^{p_i+q_j}\, A^{C(n,k)+1-i}_{C(n,k)+1-j}\right].$$

Setting

$$(i_1,\ldots,i_k) = (r'_1,\ldots,r'_k), \qquad (j_1,\ldots,j_k) = (s'_1,\ldots,s'_k),$$

we have, by (1.6.10),

$$G'_{n,k}A^{(k)}G_{n,k} = \left[(-)^{(r'_1+\cdots+r'_k)+(s'_1+\cdots+s'_k)}\, A^{r}_{\underset{\sim}{s}}\right],$$

where

$$\underset{\sim}{A}^r_s = \begin{vmatrix} a^{r'_1}_{s'_1} & \cdot & a^{r'_1}_{s'_k} \\ \cdot & \cdot & \cdot \\ a^{r'_k}_{s'_1} & \cdot & a^{r'_k}_{s'_k} \end{vmatrix}$$

is the complementary minor of

$$A^r_s = \begin{vmatrix} a^{r_1}_{s_1} & \cdot & a^{r_1}_{s_{n-k}} \\ \cdot & \cdot & \cdot \\ a^{r_{n-k}}_{s_1} & \cdot & a^{r_{n-k}}_{s_{n-k}} \end{vmatrix}$$

in $\|A\|$.

Now we know that

$$(-1)^{(r'_1+\dots+r'_k)+(s'_1+\dots+s'_k)}$$

$$= (-1)^{(r'_1+\dots+r'_k+r_1+\dots+r_{n-k})+(s'_1+\dots+s'_k+s_1+\dots+s_{n-k})+p_r+p_s}$$

$$= (-1)^{\frac{1}{2}n(n+1)+\frac{1}{2}n(n+1)+p_r+p_s}$$

$$= (-1)^{p_r+p_s} \quad .$$

Hence

$$G'_{n,k}A^{(k)}G_{n,k} = \left[(-)^{p_r+p_s}\,\underset{\sim}{A}^r_s\right]$$

$$= \operatorname{adj}^{(n-k)} A \,, \qquad \text{by } 1.3.$$

Taking the transpose of both sides and using (1.4.2), we obtain the following theorem which is a generalization of that stated as Ex. 2, p. 94, [1].

(1.9.1) Theorem. If A is an $n\times n$ matrix over K, then

$$\operatorname{adj}^{(n-k)} A = G'_{n,k}\,(A')^{(k)}\,G_{n,k} \qquad 0 \leq k \leq n.$$

Several corollaries follow from this theorem.

(1.9.2) $$A^{(k)} = G_{n,k}\,(\operatorname{adj}^{(n-k)}A)'\,G'_{n,k},$$

(1.9.3) $$|A|^{C(n-1,k)-1}A^{(k)} = G_{n,k}(\operatorname{adj} A^{(n-k)})'\,G'_{n,k},$$

(1.9.4) $$|A|^{C(n-1,k)-1}A^{(k)}$$
$$= (G_{n,k}G_{C(n,k),1})\,(A^{(n-k)})^{(C(n,k)-1)}\,(G_{n,k}G_{C(n,k),1})'$$

(1.9.5) $$A^{(k)} = |A|\,G_{n,k}((A^{(n-k)})^{-1})'G'_{n,k},\quad |A| \neq 0 .$$

(1.9.6) $$A^{(k)}G_{n,k}(A^{(n-k)})' = |A|\,G_{n,k},$$

Proof of (1.9.2),...,(1.9.6):

$$G_{n,k}(\operatorname{adj}^{(n-k)}A)G'_{n,k} = G_{n,k}G'_{n,k}(A')^{(k)}G_{n,k}G'_{n,k} \qquad \text{by (1.9.1)}$$

$$= (A^{(k)})' \qquad \text{by (1.4.2) and (1.8.2)}$$

Taking the transpose of both sides of the second equation completes the proof of (1.9.2).

$$|A|^{C(n-1,k)-1}A^{(k)}$$

$$= G_{n,k}(|A|^{C(n-1,k)-1}\operatorname{adj}^{(n-k)}A)'G'_{n,k} \qquad \text{by (1.9.2)}$$

$$= G_{n,k}(\operatorname{adj} A^{(n-k)})'G'_{n,k} \quad \text{by (1.4.12)--This proves (1.9.3)}$$

$$= G_{n,k}G'_{C,C-1}G_{C,C-1}(\operatorname{adj} A^{(n-k)})'G'_{C,C-1}G_{C,C-1}G'_{n,k} \quad \text{by (1.8.2)}$$

$$= (G_{n,k}G_{C,1})(A^{(n-k)})^{(C-1)}(G_{n,k}G_{C,1})' \quad \text{by (1.9.2), (1.8.2)}$$

where $C = C(n,k)$. This proves (1.9.4).

Now if $|A| \neq 0$, we have, by (1.9.3) and (1.4.9),

$$|A|^{C(n-1,k)-1}A^{(k)} = |A|^{C(n-1,k)}G_{n,k}(|A^{(n-k)}|^{-1}\operatorname{adj} A^{(n-k)})'G'_{n,k}$$

whence

$$A^{(k)} = |A|\,G_{n,k}((A^{(n-k)})^{-1})'G'_{n,k}. \quad \text{This proves (1.9.5).}$$

$$A^{(k)}G_{n,k} = G_{n,k}(\operatorname{adj}^{(n-k)}A)' \qquad \text{by (1.9.2) and (1.8.2)}$$

$$A^{(k)}G_{n,k}(A^{(n-k)})' = G_{n,k}|A|\,I_{C(n,k)} \qquad \text{by (1.4.7)}$$

$$= |A|\,G_{n,k}. \qquad \text{This proves (1.9.6).}$$

Now, in view of (1.9.6), we have the following theorem.

(1.9.7) <u>Theorem.</u> If A is an $n \times n$ matrix over K, then

(1) $A^{(k)} G_{n,k} (A^{(n-k)})' = (A^{(k)})' G_{n,k} A^{(n-k)} = |A| \, G_{n,k}$,

(2) $A^{(n-k)} G_{n,n-k} (A^{(k)})' = (A^{(n-k)})' G_{n,n-k} A^{(k)} = |A| \, G_{n,n-k}$.

Chapter 2

Determinants Whose Elements are Minors of Others

In this chapter we concern ourselves with several determinant identities which will be useful later. We have already mentioned two such identities, namely, (1.4.9) and (1.4.10). Others are to be found in [1], [9, pp. 71-101], and [11]. First, however, we must have some more notation.

<u>2.1. Notation.</u> Let $A = \left[a^i_j\right]$, $i = 1,\dots,m$, $j = 1,\dots,n$, be given. We adopt the following symbols.

(2.1.1) $[A]^{i_1\dots i_h}$, $1 \leq h \leq m$, will denote the $h \times n$ submatrix of A formed by the h rows designated by the integers $i_1,\dots,i_h$. Similarly, $[A]_{j_1\dots j_k}$, $1 \leq k \leq n$ will denote the $m \times k$ submatrix consisting of the k columns of A designated by $j_1,\dots,j_k$. Finally, $[A]^{i_1\dots i_h}_{j_1\dots j_k}$ will denote the $h \times k$ submatrix of A contained in both $[A]^{i_1\dots i_h}$ and $[A]_{j_1\dots j_k}$. In particular, $[A]^i_j = a^i_j$ and $[A]^{1\dots m}_{1\dots n} = A$.

(2.1.2) $A^{i_1\dots i_h}_{j_1\dots j_h}$ will denote the determinant of $[A]^{i_1\dots i_h}_{j_1\dots j_h}$.

When A is square, $\widetilde{A^{i_1 \dots i_h}_{j_1 \dots j_h}}$ will denote the signed complement in $|A|$ of the minor $A^{i_1 \dots i_h}_{j_1 \dots j_h}$ and, in particular, $\tilde{a}^i_j$ will denote the cofactor of a^i_j in $|A|$.

(2.1.3) When A is square, $\tilde{A}$ will denote the transposed adjoint matrix of A.

(2.1.4) $\{i_1 < \dots < i_h\} \cup \{j_1 < \dots < j_k\} = \{1, \dots, h+k\}$ will be used to express both the fact that $i_1 < \dots < i_h$, $j_1 < \dots < j_k$ and the fact that the set $\{i_1, \dots, i_h, j_1, \dots, j_k\}$ is some permutation of the first $h+k$ natural numbers.

$i/i_1 \dots i_k/S(n,k)$ will be used to express the fact that $(i_1, \dots, i_k)$ is the i-th term of the sequence (1.6.1).

<u>2.2. Some Well Known Theorems.</u> Since the proofs of the following theorems are readily available elsewhere, they will be omitted here.

(2.2.1) <u>Jacobi's Theorem (1834).</u> Any minor of order k in adj A is equal to the complementary signed minor in A', multiplied by $|A|^{k-1}$, see [1, pp. 97-99].

(2.2.2) <u>Franke's Theorem.</u> Any minor of order h in the k-th compound of an $n \times n$ matrix A is equal to the complementary signed minor in the k-th adjoint compound transposed matrix of A, multiplied by $|A|^{h-C(n-1,k)}$, see [1, pp. 100-101].

(2.2.3) Theorem. Let $A = \left[a^i_j\right]$ be an nxn matrix over K and let

$$D(x) = \begin{vmatrix} a^1_1+x & a^1_2 & \cdot & a^1_n \\ a^2_1 & a^2_2+x & \cdot & a^2_n \\ \cdot & \cdot & \cdot & \cdot \\ a^n_1 & a^n_2 & \cdot & a^n_n+x \end{vmatrix}.$$

Then

$$D(x) = x^n + S_1x^{n-1} + \dots + S_n.$$

where S_k is the sum of all k-rowed principal minors of $|A|$, see [9, pp. 114-115]. In particular, therefore, $S_n = |A|$.

2.3. Sylvester's Theorem on Superdeterminants. Let $A = \left[a^i_j\right]$ be any nxn matrix over K and let $\left[A\right]^{r_1\dots r_h}_{s_1\dots s_h}$, $0 \leqq h \leqq n-1$, be an arbitrary, but fixed, hxh submatrix of A. Let B be the submatrix of $A^{(h+k)}$, $1 \leqq k \leqq n-h$, which consists of those, and only those, elements of $A^{(h+k)}$ which contain $A^{r_1\dots r_h}_{s_1\dots s_h}$ as a minor. Then $B = \left[b^u_v\right]$ is clearly a square matrix of order $C(n-h,k)$. Moreover, we have, by 1.3,

(2.3.1) $$b^u_v = \left[A^{(h+k)}\right]^{i_u}_{j_v} = A^{(i_u)_1\dots(i_u)_{h+k}}_{(j_v)_1\dots(j_v)_{h+k}},$$

$u,v = 1,\dots,C(n-h,k)$.

Let r'_α, s'_α, u_β, v_β, $\alpha = 1,\ldots,n-h$, $\beta = 1,\ldots,k$, be such that

(2.3.2) $\{r_1 < \cdots < r_h\} \cup \{r'_1 < \cdots < r'_{n-h}\} = \{1,\ldots,n\}$

$= \{s_1 < \cdots < s_h\} \cup \{s'_1 < \cdots < s'_{n-h}\}$ and

$\{r_1 < \cdots < r_h\} \cup \{r'_{u_1} < \cdots < r'_{u_k}\} = \{(i_u)_1 < \cdots < (i_u)_{h+k}\}$

$\{s_1 < \cdots < s_h\} \cup \{s'_{v_1} < \cdots < s'_{v_k}\} = \{(j_v)_1 < \cdots < (j_v)_{h+k}\}$

Since

$i_u/(i_u)_1 \cdots (i_u)_{h+k}/S(n,h+k)$ and $j_v/(j_v)_1 \cdots (j_v)_{h+k}/S(n,h+k)$,

by 1.5, it follows from (2.3.2) that

(2.3.2) $u/u_1 \cdots u_k/S(n-h,k)$ and $v/v_1 \cdots v_k/S(n-h,k)$.

Let u'_α and v'_α, $\alpha = 1,\ldots,n-h-k$, be such that

(2.3.3) $\{r'_{u_1} < \cdots < r'_{u_k}\} \cup \{r'_{u'_1} < \cdots < r'_{u'_{n-h-k}}\} = \{r'_1 < \cdots < r'_{n-h}\}$

and $\{s'_{v_1} < \cdots < s'_{v_k}\} \cup \{s'_{v'_1} < \cdots < s'_{v'_{n-h-k}}\} = \{s'_1 < \cdots < s'_{n-h}\}$.

Finally let

(2.3.4) $p_u = \sum_{t=1}^{k} u_t$, $p_v = \sum_{t=1}^{k} v_t$, $p_r = \sum_{t=1}^{h} r_t$,

$p_s = \sum_{t=1}^{h} s_t$, $p_{r'} = \sum_{t=1}^{n-h} r'_t$, $p_{s'} = \sum_{t=1}^{n-h} s'_t$, $p_{r'_u} = \sum_{t=1}^{k} r'_{u_t}$,

$p_{s'_v} = \sum_{t=1}^{k} s'_{v_t}$, $p_{r'_{u'}} = \sum_{t=1}^{n-h-k} r'_{u'_t}$, and $p_{s'_{v'}} = \sum_{t=1}^{n-h-k} s'_{v'_t}$.

It follows from (2.3.2) and (2.3.3) that

$$(2.3.5)\quad \{u_1 < \cdots < u_k\} \cup \{u'_1 < \cdots < u'_{n-h-k}\} = \{1,\ldots,n-h\}$$
$$= \{v_1 < \cdots < v_k\} \cup \{v'_1 < \cdots < v'_{n-h-k}\}$$

and

$$(2.3.6)\quad p_r + p_{r'} = p_r + p_{r'_u} + p_{r'_{u'}} = \tfrac{1}{2}n(n+1) \quad \text{and}$$

$$p_s + p_{s'} = p_s + p_{s'_v} + p_{s'_{v'}} = \tfrac{1}{2}n(n+1).$$

By Jacobi's theorem (2.2.1) and in view of (2.3.2), (2.3.3) we have, from (2.3.1),

$$\left\| [\tilde{A}]^{r'_{u'_1}\cdots r'_{u'_{n-h-k}}}_{s'_{v'_1}\cdots s'_{v'_{n-h-k}}} \right\| = (-1)^{p_{r'_{u'}}+p_{s'_{v'}}} \|A\|^{n-h-k-1} b^u_v.$$

Multiplying both members of this equation by $(-1)^{p_u+p_v}$, we obtain, for the left member, in view of (2.1.2), (2.1.3), (2.3.4), and (2.3.5),

$$\overbrace{\left[[\tilde{A}]^{r'_1\cdots r'_{n-h}}_{s'_1\cdots s'_{n-h}} \right]}^{}{}^{u_1\cdots u_k}_{v_1\cdots v_k} = \left[\left[\operatorname{adj}^{(k)} [\tilde{A}]^{r'_1\cdots r'_{n-h}}_{s'_1\cdots s'_{n-h}} \right]' \right]^u_v ;$$

hence, we have

$$\left[\left[\operatorname{adj}^{(k)} [\tilde{A}]^{r'_1\cdots r'_{n-h}}_{s'_1\cdots s'_{n-h}} \right]' \right]^u_v = (-1)^{p_{r'_{u'}}+p_{s'_{v'}}+p_u+p_v} \|A\|^{n-h-k-1} b^u_v.$$

Therefore

$$(2.3.7)\quad \left[\operatorname{adj}^{(k)} [\tilde{A}]^{r'_1\cdots r'_{n-h}}_{s'_1\cdots s'_{n-h}} \right]' = \|A\|^{n-h-k-1} \left[(-1)^q \, b^u_v \right],$$

where $q = p_{r'_{u'}}+p_{s'_{v'}}+p_u+p_v$; $u,v = 1,\ldots,C(n-h,k)$.

Taking determinants of both sides of (2.3.7), we have, in view of (1.4.10),

$$\left| [\tilde{A}]^{r'_1 \ldots r'_{n-h}}_{s'_1 \ldots s'_{n-h}} \right|^{C(n-h-1,k)} = \|A\|^{Q} \left| (-1)^{q} \, b^{u}_{v} \right| ,$$

where $Q = (n-h-k-1)C(n-h,k)$ and $q = p_{r'_{u'}} + p_{s'_{v'}} + p_u + p_v$.

Now if we apply Jacobi's theorem (2.2.1) to the determinant on the left and factor

$$(-1)^{p_{r'_{u'}}+p_u} \qquad \text{and} \qquad (-1)^{p_{s'_{v'}}+p_v}$$

respectively out of the elements of the u-th row and the v-th column, $u,v = 1,\ldots,C(n-h,k)$, of the second determinant on the right, the last equation becomes

$$\left((-1)^{p_{r'}+p_{s'}} \|A\|^{n-h-1} A^{r_1 \ldots r_h}_{s_1 \ldots s_h} \right)^{C(n-h-1,k)} = (-1)^{\Sigma_1+\Sigma_2} \|A\|^{Q} \|B\| ,$$

where

$$\Sigma_1 = \sum_{u=1}^{C(n-h,k)} (p_{r'_{u'}} + p_u)$$

$$= \sum_{u=1}^{C(n-h,k)} (r'_{u'_1} + \ldots + r'_{u'_{n-h-k}}) + \sum_{u=1}^{C(n-h,k)} (u_1 + \ldots + u_k) \qquad \text{by (2.3.4)}$$

$$= (r'_1 + \ldots + r'_{n-h})C(n-h-1,n-h-k-1) \qquad \text{by (2.3.2),(2.3.3),1.6}$$
$$+ (1 + \ldots + n-h)C(n-h-1,k-1) \qquad \text{by (2.3.2)}$$

$$= p_{r'}C(n-h-1,k) + \tfrac{1}{2}n(n-h)(n-h+1)C(n-h-1,k-1) \qquad \text{by (2.3.4)}$$

and, similarly,

$$\Sigma_2 = p_{s'}C(n-h-1,k)+\tfrac{1}{2}(n-h)(n-h+1)C(n-h-1,k-1).$$

Now, since $(n-h)(n-h+1)$ is an even integer, we have

$$(-1)^P\left(|A|^{n-h-1}A^{r_1\dots r_h}_{s_1\dots s_h}\right)^{C(n-h-1,k)} = (-1)^P|A|^Q|B| ,$$

where $P = (p_{r'}+p_{s'})C(n-h-1,k)$. Since this is a polynomial identity, we obtain, in view of the fact that

$$\begin{aligned}(n-h-1)C(n-h-1,k)-Q &= (n-h-1)C(n-h-1,k)-(n-h-k-1)C(n-h,k)\\ &= C(n-h-1,k-1) \quad ,\end{aligned}$$

$$|A|^{C(n-h-1,k-1)}\left(A^{r_1\dots r_h}_{s_1\dots s_h}\right)^{C(n-h-1,k)} = |B| \quad .$$

Therefore we have the following theorem.

(2.3.8) <u>Theorem.</u> If A is any nxn matrix over K, if $A^{r_1\dots r_h}_{s_1\dots s_h}$, $0 \leq h \leq n-1$ is an arbitrary, but fixed, minor of $|A|$, and if B is the submatrix of $A^{(h+k)}$, $1 \leq k \leq n-h$, which consists of those, and only those, elements of $A^{(h+k)}$ which contain $A^{r_1\dots r_h}_{s_1\dots s_h}$ as a minor, then

$$|B| = |A|^{C(n-h-1,k-1)}\left(A^{r_1\dots r_h}_{s_1\dots s_h}\right)^{C(n-h-1,k)} \quad .$$

In particular, when $r_\alpha = s_\alpha = \alpha$, $\alpha = 1,\ldots,h$, (2.3.8) becomes Sylvester's theorem on superdeterminants, see [11, p. 85].

2.4. Bordered Determinants and Matrices.

(2.4.0) <u>Definition.</u> Let the $(n+1)x(n+1)$ matrices $\left(1 \quad \check{x}^i_j\right)$ and $\left(1 \quad \hat{x}^i_j\right)$ respectively be derived from

$$\left(1 \quad x^i_j\right) = \begin{bmatrix} 1 & x^0_1 & \cdot & x^0_n \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^n_1 & \cdot & x^n_n \end{bmatrix}$$

by replacing x^i_j by $\check{x}^i_j = -\tilde{x}^{n-i}_{n+1-j}$ and by $\hat{x}^i_j = (-)^{i+j+1}\tilde{x}^{n-i}_{n+1-j}$, where $\tilde{x}^{n-i}_{n+1-j}$ is the cofactor of x^{n-i}_{n+1-j} in $\left\| 1 \quad x^i_j \right\|$. Then it is true that

$$(2.4.1) \qquad \left\| 1 \quad \check{x}^i_j \right\| = (n+1) \left\| 1 \quad x^i_j \right\|^{n-1} \qquad \text{and}$$

$$(2.4.2) \qquad \left\| 1 \quad \hat{x}^i_j \right\| = \left\| 1 \quad x^i_j \right\|^{n-1} \quad \text{or} \quad 0$$

according as n is an even or an odd integer.

To prove (2.4.1), we expand $\left\| 1 \quad \check{x}^i_j \right\|$ in terms of the elements of its first column to obtain, in the notation of (2.1.1),

$$\left\| 1 \quad \check{x}^i_j \right\| = \sum_{i=0}^{n} (-1)^i \left\| \left(1 \quad \check{x}^i_j\right)^{0\ldots(i-1)(i+1)\ldots n}_{1\ldots n} \right\| .$$

Then we reverse the orders of the rows and of the columns of each of the determinants in the right member and use

$\check{x}^i_j = -\tilde{x}^{n-i}_{n+1-j}$ to obtain, using (2.1.3) and (2.2.1),

$$\left\| 1 \quad \check{x}^i_j \right\| = \sum_{i=0}^{n} (-1)^i \left\| \left[-\widetilde{\left[1 \quad x^i_j\right]} \right]^{0\ldots(n-i-1)(n-i+1)\ldots n}_{1\ldots n} \right\| ,$$

$$= \sum_{i=0}^{n} (-1)^{n+i} (-1)^{n-i} \left\| 1 \quad x^i_j \right\|^{n-1}$$

$$= (n+1) \left\| 1 \quad x^i_j \right\|^{n-1} .$$

The proof is complete.

Similarly, to prove (2.4.2), we expand $\left\| 1 \quad \hat{x}^i_j \right\|$ in terms of the elements of its first column to obtain

$$\left\| 1 \quad \hat{x}^i_j \right\| = \sum_{i=0}^{n} (-1)^i \left\| \left[1 \quad \hat{x}^i_j \right]^{0\ldots(i-1)(i+1)\ldots n}_{1\ldots n} \right\| .$$

Since, in view of 1.2, 1.3, and (2.1.1),

$$\hat{x}^i_j = (-1)^{i+j+1} \tilde{x}^{n-i}_{n+1-j}$$

$$= (-1)^{i+j+1} (-1)^{2n+1-i-j} \left[\left[1 \quad x^i_j \right]^{(n)} \right]^i_{j-1}$$

we have, by (2.2.2) and 1.3,

$$\left\| 1 \quad \hat{x}^i_j \right\| = \sum_{i=0}^{n} (-1)^i \left\| \left[\left[1 \quad x^i_j \right]^{(n)} \right]^{0\ldots(i-1)(i+1)\ldots n}_{0\ldots(n-1)} \right\|$$

$$= \sum_{i=0}^{n} (-1)^i (-1)^{n+i} \left[\operatorname{adj}^{(n)} \begin{bmatrix} 1 & x_j^i \end{bmatrix}' \right]_n^i \begin{vmatrix} 1 & x_j^i \end{vmatrix}^{n-1}$$

$$= \sum_{i=0}^{n} (-1)^i (-1)^{n+1} (-1)^{n+i} 1 \cdot \begin{vmatrix} 1 & x_j^i \end{vmatrix}^{n-1}$$

$$= \begin{vmatrix} 1 & x_j^i \end{vmatrix}^{n-1} \sum_{i=0}^{n} (-1)^i$$

$$= \begin{vmatrix} 1 & x_j^i \end{vmatrix}^{n-1} \quad \text{or} \quad 0$$

according as n is an even or an odd integer. This completes the proof.

Let $\begin{bmatrix} A & X' \\ Y & 0 \end{bmatrix}$, $\begin{bmatrix} B & U' \\ V & 0 \end{bmatrix}$ be $(m+n)\times(m+n)$ matrices obtained by bordering the $n\times n$ matrices A, B on the right by the transposes of the $m\times n$ matrices X, U, below by the $m\times n$ matrices Y, V, and at the lower right corner by the $m\times m$ zero matrix, respectively, where b_j^i is the cofactor of a_j^i in $\begin{vmatrix} A & X' \\ Y & 0 \end{vmatrix}$, $i,j = 1,\dots,n$. Then it is true that

$$(2.4.3) \qquad \begin{vmatrix} B & U' \\ V & 0 \end{vmatrix} = \begin{vmatrix} U & X' \end{vmatrix} \begin{vmatrix} V & Y' \end{vmatrix} \begin{vmatrix} A & X' \\ Y & 0 \end{vmatrix}^{n-m-1} \quad \text{or} \quad 0$$

according as $m \leqq n$ or $m > n$.

If $m > n$, (2.4.3) is seen to be true if one expands the determinant in the left member in terms of the minors of its last m rows(or columns) by Laplace's method. If $m = n$, we again expand, by Laplace's method, in terms of the last m rows (or columns) to obtain

$$\begin{vmatrix} B & U' \\ V & O \end{vmatrix} = |U'|\,|V| = |U|\,|V|.$$

On the other hand, we have, when $m = n$,

$$|U\ X'|\,|V\ Y'|\,\begin{vmatrix} A & X' \\ Y & O \end{vmatrix}^{n-n-1} = |U|\,|X'|\,|V|\,|Y'|\,|X'|^{-1}|Y|^{-1} = |U|\,|V|.$$

Which shows that (2.4.3) is true for $m = n$, also. If $m < n$, we expand, by Laplace's method, in terms of the minors of the last m columns to obtain

$$\begin{vmatrix} B & U' \\ V & O \end{vmatrix} = \sum_{i=1}^{C(n,m)} (-1)^{r_i}\, U_i \begin{vmatrix} [B]^{i_1'\cdots i_{n-m}'} \\ V \end{vmatrix},$$

where $i/i_1\cdots i_m/S(n,m)$, $\{i_1 < \cdots < i_m\} \cup \{i_1' < \cdots < i_{n-m}'\}$ $= \{1,\ldots,n\}$, $r_i = i_1+\cdots+i_m + \frac{1}{2}m(2n+m+1)$, and $U_i = \left|[U']^{i_1\cdots i_m}\right|$, in accordance with (2.1.1) and (2.1.4). Then we expand the determinants appearing as the last factor in the terms of the above equation by Laplace's method, using the last m rows, to obtain

$$\begin{vmatrix} B & U' \\ V & O \end{vmatrix} = \sum_{i=1}^{C(n,m)} (-1)^{r_i}\, U_i \sum_{j=1}^{C(n,m)} (-1)^{s_j}\, V_j\, B_{j_1'\cdots j_{n-m}'}^{i_1'\cdots i_{n-m}'},$$

where $j/j_1\cdots j_m/S(n,m)$, $\{j_1 < \cdots < j_m\} \cup \{j_1' < \cdots < j_{n-m}'\}$ $= \{1,\ldots,n\}$, $s_j = j_1+\cdots+j_m+\frac{1}{2}m(2n-m+1)$, and $V_j = \left|[V]_{j_1\cdots j_m}\right|$ in accordance with (2.1.1), (2.1.2), and (2.1.4). Next, since

$$B_{j_1'\cdots j_{n-m}'}^{i_1'\cdots i_{n-m}'} = \left| \left[\widetilde{\begin{bmatrix} A & X' \\ Y & 0 \end{bmatrix}} \right]_{j_1'\cdots j_{n-m}'}^{i_1'\cdots i_{n-m}'} \right| \qquad \text{by hypothesis}$$

$$= (-1)^{p_{i'}+p_{j'}} \left| \begin{matrix} [A]_{j_1\cdots j_m}^{i_1\cdots i_m} & [X']^{i_1\cdots i_m} \\ [Y]_{j_1\cdots j_m} & 0 \end{matrix} \right| \left| \begin{matrix} A & X' \\ Y & 0 \end{matrix} \right|^{n-m-1}$$

$$= (-1)^{p_{i'}+p_{j'}+m(2m+1)} X_i Y_j \left| \begin{matrix} A & X' \\ Y & 0 \end{matrix} \right|^{n-m-1} \qquad \text{by (2.2.1),}$$

where $p_{i'} + p_{j'} = (i_1'+\cdots+i_{n-m}') + (j_1'+\cdots+j_{n-m}')$,

$X_i = \left| [X']^{i_1\cdots i_m} \right|$ and $Y_j = \left| [Y]_{j_1\cdots j_m} \right|$, we have

$$\left| \begin{matrix} B & U' \\ V & 0 \end{matrix} \right| = \sum_{i=1}^{C(n,m)} (-1)^{r_i} U_i \sum_{j=1}^{C(n,m)} (-1)^{s_j} V_j (-1)^{r_i'+s_j'} X_i Y_j \left| \begin{matrix} A & X' \\ Y & 0 \end{matrix} \right|^{n-m-1}$$

$$= \sum_{i=1}^{C(n,m)} (-1)^{r_i+r_i'} U_i X_i \sum_{j=1}^{C(n,m)} (-1)^{s_j+s_j'} V_j Y_j \left| \begin{matrix} A & X' \\ Y & 0 \end{matrix} \right|^{n-m-1},$$

where $r_i' = p_{i'} + \frac{1}{2}m(2m+1)$, $s_j' = p_{j'} + \frac{1}{2}m(2m+1)$.

We know that

$$r_i+r_i' = \sum_{t=1}^{m} i_t + \sum_{t=1}^{n-m} i_t' + \tfrac{1}{2}m(2n+m+1) + \tfrac{1}{2}m(2m+1)$$

$$= \tfrac{1}{2}n(n+1) + \tfrac{1}{2}m(2n+m+1) + \tfrac{1}{2}m(2m+1)$$

and, similarly,

$$s_j + s_j' = \tfrac{1}{2}n(n+1) + \tfrac{1}{2}m(2n-m+1) + \tfrac{1}{2}m(2m+1).$$

Therefore we have, for $P = r_i + r_i' + s_j + s_j'$,

$$\begin{Vmatrix} B & U' \\ V & 0 \end{Vmatrix} = (-1)^P \left(\sum_{i=1}^{C(n,m)} U_i\ X_i\right)\left(\sum_{j=1}^{C(n,m)} V_j\ Y_j\right)\begin{Vmatrix} A & X' \\ Y & 0 \end{Vmatrix}^{n-m-1}$$

$$= (-1)^P \begin{Vmatrix} U\ X' \end{Vmatrix} \begin{Vmatrix} V\ Y' \end{Vmatrix} \begin{Vmatrix} A & X' \\ Y & 0 \end{Vmatrix}^{n-m-1}$$

by the Binet-Cauchy multiplication theorem, see [11, pp. 77-78]. Then, since

$$P = n(n+1) + 2mn + m(2m+1) + m$$
$$= n(n+1) + 2mn + 2m(m+1)$$

is an even integer, we have (2.4.3) for $m < n$. This completes the proof for all possible cases.

Setting $X = Y = U = V = [1,\ldots,1]$ in (2.4.3), we have $m = 1$ and $\|U\ X'\| = \|U\ Y'\| = \sum_1^n 1 = n$; hence the next theorem follows as a corollary.

(2.4.4) <u>Theorem.</u> If each element a_j^i, $i,j = 1,\ldots,n$, be replaced by its cofactor $\tilde{a}_j^i$ in

$$\begin{Vmatrix} a & 1 \\ 1 & 0 \end{Vmatrix} = \begin{Vmatrix} a_1^1 & \cdot & a_n^1 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ a_1^n & \cdot & a_n^n & 1 \\ 1 & \cdot & 1 & 0 \end{Vmatrix},$$

then, for the derived determinant, we have

$$\begin{vmatrix} \tilde{a} & 1 \\ 1 & 0 \end{vmatrix} = n^2 \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}^{n-2} , \quad n \geq 1 .$$

We next prove a theorem which is a sort of dual to (2.4.4), namely,

(2.4.5) Theorem. Let the $(n+1) \times (n+1)$ matrix $\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$ be given and let each element a^i_j, $i,j = 1,\ldots,n$, be replaced by

$$\hat{a}^i_j = \left[\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}^{(n)}\right]^{i+1}_{j+1} ,$$

the element in the $(i+1)$-th row and $(j+1)$-th column of the n-th compound of $\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$. We thus obtain a new $(n+1)x(n+1)$ matrix, which we denote by $\begin{bmatrix} \hat{a} & 1 \\ 1 & 0 \end{bmatrix}$, for which it is true that

$$\begin{vmatrix} \hat{a} & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}^{n-2} \quad \text{or} \quad 0$$

according as n is an odd or an even integer.

We prove (2.4.5) by first expanding $\begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}$ in terms of the elements, and their cofactors, of its last column, then expanding each of the cofactors in terms of the elements of its last row to obtain

$$\begin{vmatrix} \hat{a} & 1 \\ 1 & 0 \end{vmatrix} = \sum_{i,j=1}^{n} (-1)^{i+j+1} \hat{a}^{1\ldots(i-1)(i+1)\ldots n}_{1\ldots(j-1)(j+1)\ldots n}$$

$$= \sum_{i,j=1}^{n} (-1)^{i+j+1} \left| \left[\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}^{(n)}\right]^{2\ldots i(i+2)\ldots(n+1)}_{2\ldots j(j+2)\ldots(n+1)} \right| .$$

Applying (2.2.2) to this, we obtain

$$\begin{vmatrix} \hat{a} & 1 \\ 1 & 0 \end{vmatrix} = \sum_{i,j=1}^{n} (-1)^{i+j+1} (-1)^{i+j} \left| \left[\left(\operatorname{adj}^{(n)} \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} \right)' \right]^{1\ (i+1)}_{1\ (j+1)} \right| \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}^{n-2}$$

$$= \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}^{n-2} \sum_{i,j=1}^{n} - \begin{vmatrix} 0 & (-1)^{j} \\ (-1)^{i} & (-)^{i+j} a^{n-i+1}_{n-j+1} \end{vmatrix}$$

$$= \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}^{n-2} \sum_{i,j=1}^{n} (-1)^{i+j} = \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}^{n-2} \quad \text{or} \quad 0$$

according as n is odd or even. This completes the proof.

More generally, let the $(n+1)x(n+1)$ matrices

$$\begin{bmatrix} \tilde{a}_{(k)} & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{a}_{(k)} & 1 \\ 1 & 0 \end{bmatrix}$$

be derived from the $(n+1)x(n+1)$ matrix

$$(2.4.6) \qquad \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$$

in the following manner. Let $\hat{a}_{(k)}$ be the NxN submatrix of the $(k+1)$-th compound of (2.4.6) consisting of those, and only those, elements which are $(k+1)$-rowed minors of (2.4.6) having in their lower right corners the lower right corner element of (2.4.6), namely, zero. Clearly, in this instance, $N = C(n,k)$. Similarly, let $\tilde{a}_{(k)}$ be the NxN submatrix of the k-th adjoint compound transposed matrix of (2.4.6) consisting of those elements, and only those, which are signed (n-k+1)-rowed minors of (2.4.6) having in their lower right corners the lower right corner element of (2.4.6). Again, it is evident that $N = C(n,k)$. Then let $\tilde{a}_{(k)}$ and $\hat{a}_{(k)}$ each be bordered on the right with a column of 1's, below with a row of 1's, and on the lower right corner with zero. Then it is true that

$$(2.4.7) \qquad \begin{vmatrix} \tilde{a}_{(k)} & 1 \\ 1 & 0 \end{vmatrix} = n^2 \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}^{n-k-1}, \text{ or } -1, \text{ or } 0$$

according as $k = 1 < n$, or $k = n$, or $1 < k < n$; and

$$(2.4.8) \qquad \begin{vmatrix} \hat{a}_{(k)} & 1 \\ 1 & 0 \end{vmatrix} = 0, \text{ or } -1, \text{ or } \frac{1-(-1)^n}{2} \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}^{k-1}$$

according as $k = 1 < n$, or $k = n$, or $1 < k < n$.

In case $k = 1 < n$, (2.4.7) is just a restatement of (2.4.4) and, hence, has already been proved. Similarly, for $1 < k < n$, (2.4.8) is just a restatement of (2.4.5) and, hence, needs no further proof. Moreover, we have

$$\begin{vmatrix} \tilde{a}_{(n)} & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

and

$$\begin{vmatrix} \hat{a}_{(n)} & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix} & 1 \\ 1 & 0 \end{vmatrix} = -1 ,$$

which shows that both (2.4.7) and (2.4.8) are true when

$k = n$. The fact that (2.4.7) is true when $1 < k < n$ and (2.4.8) is true when $k = 1 < n$ follows from the considerations of the next section. Hence (2.4.7) and (2.4.8) are true in all cases.

2.5. More General Considerations.

Let

$$E = \begin{bmatrix} A & X' \\ Y & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \hat{A}_{(k)} & U' \\ V & W \end{bmatrix}$$

be partitioned $(n+m)x(n+m)$ and $(N+M)x(N+M)$ matrices, respectively, where A is a square matrix of order n, X and Y are mxn matrices, 0 is the mxm zero matrix, U and V are any MxN matrices, W is any MxM matrix, but $\hat{A}_{(k)}$ is the square matrix of order $N = C(n,k)$ derived from E by taking, see (2.1.1),

$$\left[\hat{A}_{(k)}\right]^i_j = \begin{vmatrix} [A]^{i_1\ldots i_k}_{j_1\ldots j_k} & [X']^{i_1\ldots i_k} \\ [Y]_{j_1\ldots j_k} & [0]^{1\ldots m}_{1\ldots m} \end{vmatrix},$$

$i/i_1\ldots i_k/S(n,k)$ and $j/j_1\ldots j_k/S(n,k)$, see (2.1.4). Now it is easy to verify that the relation

$$(2.5.1)\quad \left[\hat{A}_{(k)}\right]^i_j = (-1)^{p_j} \begin{vmatrix} [A]^{i_1\ldots i_k}_{1\ldots n} & [X']^{i_1\ldots i_k} \\ [Y] & [0]^{1\ldots m}_{1\ldots m} \\ [I_n]^{j'_1\ldots j'_{n-k}}_{1\ldots n} & [0]^{1\ldots n-k}_{1\ldots m} \end{vmatrix},$$

where $\{j_1 < \cdots < j_k\} \cup \{j'_1 < \cdots < j'_{n-k}\} = \{1,\ldots,n\}$, I_n is the $n \times n$ unit matrix, and

$$p_j = \tfrac{1}{2}(n-k)(2m+n+k+1) + \sum_{t=1}^{n-k} j'_t ,$$

is true. Now let

$$j = N'' + \alpha, \quad = 1,\ldots,N', \quad N'' = C(n-1,k-1), \quad N' = C(n-1,k).$$

From the considerations in 1.6, we see that

$$(j_\alpha)'_1 = 1 \quad \text{for} \quad \alpha = 1,\ldots,N'.$$

For each α and fixed i there are clearly k elements $\left[\hat{A}_{(k)}\right]^i_j$ for which the set $\{(j_{\alpha_u})'_1 < \cdots < (j_{\alpha_u})'_{n-k}\}$ contains the set $\{(j_\alpha)'_2 < \cdots < (j_\alpha)'_{n-k}\}$ but not the integer $(j_\alpha)'_1 = 1$. Evidently each such set must contain exactly one integer $(j_\alpha)_u$ outside the set $\{(j_\alpha)'_1 < \cdots < (j_\alpha)'_{n-k}\}$ and, hence, in its complement $\{(j_\alpha)_1 < \cdots < (j_\alpha)_k\}$. In other words, the elements $\left[A_{(k)}\right]^i_j$ are such that

$$\{(j_{\alpha_u})'_1 < \cdots < (j_{\alpha_u})'_{n-k}\} = \{(j_\alpha)'_2 < \cdots < (j_\alpha)'_{n-k}\} \cup \{(j_\alpha)_u\}.$$

Let t_u be the natural number for which

$$(j_{\alpha_u})'_{t_u} = (j_\alpha)_u.$$

Then, in view (2.5.1), we have $\left[\hat{A}_{(k)}\right]^i_{j_{\alpha_u}} = D$, where

$$D = (-)^{p_{j_{\alpha_u}}} \begin{vmatrix} [A]^{i_1 \cdots i_k} & [X']^{i_1 \cdots i_k} \\ Y & [0]^{1 \cdots m}_{1 \cdots m} \\ [I_n]^{(j_\alpha)'_2 \cdots (j_\alpha)'_{t_u - 1} (j_\alpha)_u (j_\alpha)'_{t_u+1} \cdots (j_\alpha)'_{n-k}} & [0]^{1 \cdots n-k}_{1 \cdots m} \end{vmatrix} .$$

Permuting rows, we obtain

$$(2.5.2) \qquad \left[\hat{A}_{(k)}\right]^{i}_{j_{\alpha_u}}$$

$$= (-1)^{p_{j_{\alpha_u}} + t_u} \begin{vmatrix} [A]^{i_1 \cdots i_k} & [X']^{i_1 \cdots i_k} \\ Y & [0]^{1 \cdots m}_{1 \cdots m} \\ [I_n]^{(j_\alpha)_u (j_\alpha)'_2 \cdots (j_\alpha)'_{n-k}} & [0]^{1 \cdots n-k}_{1 \cdots m} \end{vmatrix} .$$

Let v be any one of the integers $1, 2, \ldots, m$, subject only to the restriction $[Y]^v_1 \neq 0$. Then, if we multiply each element of the j_α-th column of F by $[Y]^v_1$ and augment the result by

$$(-1)^{p_{j_{\alpha_u}} + p_{j_\alpha} + t_u} [Y]^v_{(j_\alpha)_u}$$

times the corresponding element of the j_{α_u}-th column, for $u = 1, 2, \ldots, k$, successively, we obtain a new matrix, call it F_α, which differs from F only in the elements of the j_α-th column. Moreover,

$$|F| = |F_\alpha| / [Y]^v_1 .$$

Now, since the transformation T_α which carries F into F_α is entirely independent of the columns for which $j = N'' + \beta$, $1 \leq \beta \neq \alpha = N'$, we can transform F_α into a new matrix $F_{\alpha\beta}$ by a like transformation T_β, $\beta \neq \alpha$. Moreover, it is clear that

$$F_{\beta\alpha} = F_{\alpha\beta} \quad \text{and} \quad |F| = |F_\alpha| / [Y]_1^v = |F_{\alpha\beta}| / ([Y]_1^v)^2 .$$

Thus, by successive transformations, we finally obtain a new matrix $F_{1\ldots N'}$, differing from F only in the elements of the columns for which $j = N'' + \alpha$, $\alpha = 1, \ldots, N'$, and for which

$$(2.5.3) \qquad |F| = |F_{1\ldots N'}| / ([Y]_1^v)^{N'} .$$

From the foregoing we have

$$\left[F_{1\ldots N'}\right]_{j_\alpha}^{i} = [Y]_1^v \left[\hat{A}_{(k)}\right]_\alpha^i + \sum_{u=1}^{\kappa} (-1)^{P_u} [Y]_{(j_\alpha)_u}^v \left[\hat{A}_{(k)}\right]_{j_{\alpha_u}}^i .$$

where $P_u = p_{j_{\alpha_u}} + p_{j_\alpha} + t_u$. Hence, in view of (2.5.1) and (2.5.2), we have

$$(2.5.4) \qquad \left[F_{1\ldots N'}\right]_{j_\alpha}^{i}$$

$$= (-1)^{p_{j_\alpha}} [Y]_1^v \begin{vmatrix} [A]^{i_1 \ldots i_k} & [X']^{i_1 \ldots i_k} \\ Y & [0]_{1\ldots m}^{1\ldots m} \\ [I_n]^{(j_\alpha)'_1 \ldots (j_\alpha)'_{n-k}} & [0]_{1\ldots m}^{1\ldots n-k} \end{vmatrix} +$$

$$\sum_{u=1}^{k}(-1)^{p_{j_\alpha}}[Y]^{v}_{(j_\alpha)_u}\begin{vmatrix} [A]^{i_1\ldots i_k} & [X']^{i_1\ldots i_k} \\ Y & [0]^{1\ldots m}_{1\ldots m} \\ [I_n]^{(j_\alpha)_u (j_\alpha)'_2 \ldots (j_\alpha)'_{n-k}} & [0]^{1\ldots n-k}_{1\ldots m} \end{vmatrix}$$

$$i = 1,\ldots,N = C(n,k) \quad , \qquad = 1,\ldots,N' = C(n-1,k),$$

$$j = \quad + (N'' = C(n-1,k-1)) \quad .$$

Now if, in the first determinant of the right member of (2.5.4), we multiply each element of the (k+m+1)-th row by $[Y]^{v}_{1}$ and augment the result by $[Y]^{v}_{(j_\alpha)'_r}$ times the corresponding element of the (k+m+r)-th row for $r = 2,3,\ldots,n-k$, successively, we obtain a new determinant differing from the new determinants under the summation symbol, obtained by multiplying the elements of the (k+m+1)-th row of the u-th determinant under the summation symbol in (2.5.4) by $[Y]^{v}_{(j_\alpha)_u}$, only in the (k+m+1)-th row. Hence we can add to obtain

$$\left[F_{1\ldots N'}\right]^{i}_{j_\alpha} = (-1)^{p_{j_\alpha}}\begin{vmatrix} [A]^{i_1\ldots i_k} & [X']^{i_1\ldots i_k} \\ Y & [0]^{1\ldots m}_{1\ldots m} \\ [\Sigma]^{\alpha}_{1\ldots n} & [0]^{1}_{1\ldots m} \\ [I_n]^{(j_\alpha)'_2\ldots(j_\alpha)'_{n-k}} & [0]^{2\ldots n-k}_{1\ldots m} \end{vmatrix} ,$$

where $$[\Sigma]^{\alpha}_{j} = \sum_{\xi=1}^{n-k}[Y]^{v}_{(j_\alpha)'_\xi}\,\delta^{(j_\alpha)'_\xi}_{j} + \sum_{u=1}^{k}[Y]^{v}_{(j_\alpha)_u}\,\delta^{(j_\alpha)_u}_{j}$$

$$= \sum_{\eta=1}^{n}[Y]^{v}_{\eta}\,\delta^{\eta}_{j} = [Y]^{v}_{j} \quad ,$$

since $(j_\alpha)' = 1$ and

$\{(j_\alpha)_1 < \dots < (j_\alpha)_k\} \cup \{(j_\alpha)'_1 < \dots < (j_\alpha)'_{n-k}\} = \{1,\dots,n\}$.

Therefore, since the (k+v)-th and the (k+m+1)-th rows of the last determinant are identical, we see that

$$(2.5.5)\quad \left[F_{1\dots N'}\right]^{i}_{j_\alpha} = 0 \quad \text{for} \quad i = 1,\dots,N = C(n,k),$$
$$\alpha = 1,\dots,N' = C(n-1,k).$$

Recalling how we obtained $F_{1\dots N'}$, we see that

$$(2.5.6)\quad \left[F_{1\dots N'}\right]^{N+r}_{j_\alpha} = [Y]^{v}_{1}\,[V]^{r}_{j_\alpha} + \sum_{u=1}^{k} (-1)^{P_u} [Y]^{v}_{(j_\alpha)_u} [V]^{r}_{(j_\alpha)_u},$$

where $P_u = p_{j_{\alpha_u}} + p_{j_\alpha} + t_u$, $r = 1,\dots,m$.

Dividing each element of the j_α-th column of $F_{1\dots N'}$ by $[Y]^{v}_{1}$, for $\alpha = 1,\dots,N'$, we obtain a new matrix

$$\hat{F} = \begin{bmatrix} \left[\hat{A}_{(k)}\right]^{1\dots N}_{1\dots N''} & [0]^{1\dots N}_{1\dots N'} & U' \\ \left[V\ \right]_{1\dots N''} & \left[\hat{V}\right]^{1\dots M}_{1\dots N'} & W \end{bmatrix},$$

where $\left[\hat{V}\right]^{r}_{\alpha} = \left[F_{1\dots N'}\right]^{N+r}_{j_\alpha} \Big/ [Y]^{v}_{1}$, by (2.5.5) and (2.5.6), for which, in view of (2.5.3), we have $|F| = |\hat{F}|$. Now we can repeat the process used to transform F into $\hat{F}$, using rows (columns) instead of columns (rows), to transform $\hat{F}$ into a new matrix

$$(2.5.7)\qquad \hat{\check{F}} = \begin{bmatrix} [\hat{A}_{(k)}]^{1\ldots N''}_{1\ldots N''} & [0]^{1\ldots N''}_{1\ldots N'} & [U']^{1\ldots N''} \\ [0\quad]^{1\ldots N'}_{1\ldots N''} & [0]^{1\ldots N'}_{1\ldots N'} & [\hat{U}']^{1\ldots N'} \\ [V\quad]_{1\ldots N''} & [\hat{V}]^{1\ldots M}_{1\ldots N'} & W \end{bmatrix}$$

for which $|\hat{\check{F}}| = |\hat{F}| = |F|$.

Hence, to evaluate $|F|$, we need merely evaluate $|\hat{\check{F}}|$. To this end, let us expand $|\hat{\check{F}}|$ by Laplace's method, using the columns involving $[\hat{V}]^{1\ldots M}_{1\ldots N'}$. Then we obtain

(2.5.8) <u>Theorem.</u> $|F| = 0$ for $M < C(n-1,k)$;

$$|F| = (-1)^{M} |\hat{U}'|\,|\hat{V}| \cdot A^{1\ldots C(n-1,k-1)}_{1\ldots C(n-1,k-1)} \quad \text{for } M = C(n-1,k)$$

In the proof of (2.5.8), we assumed $[Y]^{v}_{1} \neq 0$ for some v, $1 \leqq v \leqq m$. Suppose the assumption not valid. Then either there is some integer j, $1 \leqq j \leqq n$, for which $[Y]^{v}_{j} \neq 0$, in which case, we could interchange columns in E so as to bring this column into first place and then make the proof in the manner demonstrated, or

$$[\hat{A}_{(k)}]^{i}_{j} = \begin{vmatrix} [A]^{i_1\ldots i_k} & [X']^{i_1\ldots i_k} \\ [0]^{1\ldots m}_{1\ldots n} & [0\;]^{1\ldots m}_{1\ldots m} \end{vmatrix},$$

in which case,

$$F = \begin{vmatrix} 0 & U' \\ V & W \end{vmatrix} = 0 \quad \text{for} \quad M < C(n,k)$$

and, in particular, for $M < C(n-1,k) \leqq C(n,k)$. Hence we see that (2.5.8) is true in all cases.

If $n > 2$, then, in case $k = 1 < n$, (2.4.8) follows from (2.5.8) as a corollary, for, in this case, $m = M = 1$ and $C(n-1,k) = C(n-1,1) = n-1$, that is, $M < C(n-1,k)$. If $n = 2$, then, in case $k = 1 < n$, we have

$$\begin{vmatrix} \hat{a}_{(2)} & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0.$$

Hence (2.4.8) has been proved for the case in which $1 = k < n$. By making a similar argument, we can prove (2.4.7) for the case in which $1 < k < n$.

Now we use (2.4.1) to obtain the next theorem.

(2.5.9) <u>Theorem.</u> Let $\begin{bmatrix} 1 & x_j^i \end{bmatrix}$ be defined over a commutative field, without characteristic, which is algebraically closed. Then

$$\begin{vmatrix} 1 & x_j^i \end{vmatrix} = B_{rn} D_{n-1}(D_{n-2}(\ldots D_{r+1}(D_r(d))\ldots)),$$

where: $0 < r < n$; $i = 0,\ldots,n$; $j = 1,\ldots,n$;

$$B_{rn} = \prod_{v=1}^{n-r} \left(\frac{1}{n-v+2} \right)^{\frac{n}{(n-v)(n-v+1)}} ;$$

$$D_{n-v}(d) = \begin{vmatrix} 1 & (-)^{i_v+j_v} d_{j_v}^{i_v} \end{vmatrix}^{\frac{1}{n-v}} ;$$

$$d_{j_v}^{i_v} = \begin{vmatrix} 1 & d^{(i_1)(i_2)\cdots(i_v)_0}_{(j_1)(j_2)\cdots(j_v)_1} & \cdot & d^{(i_1)(i_2)\cdots(i_v)_0}_{(j_1)(j_2)\cdots(j_v)_{n-v}} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & d^{(i_1)(i_2)\cdots(i_v)_{n-v}}_{(j_1)(j_2)\cdots(j_v)_1} & \cdot & d^{(i_1)(i_2)\cdots(i_v)_{n-v}}_{(j_1)(j_2)\cdots(j_v)_{n-v}} \end{vmatrix} ;$$

$((i_1)_0, \ldots, (i_1)_{n-1})$ runs over the sequence of the n-tuples of the integers $0,1,\ldots,n$; i_v, $1 < v < n$, runs over the sequence $0,1,\ldots, n-v+1$ as

$$\left((i_1)\cdots(i_v)_0, \ldots, (i_1)\cdots(i_v)_{n-v} \right)$$

runs over the sequence of the (n-v+1)-tuples of the integers

$$(i_1)\cdots(i_{v-1})_0, \ldots, (i_1)\cdots(i_{v-1})_{n-v+1} ,$$

the order being lexicographical order in all cases, and similarly for j_v, with $d^{(i_1)}_{(j_1)_v}{}^{u} = x_v^u$.

In order to facilitate the proof, we first consider

$$D_{n-1}\ (\ldots D_{n-v}(F\ D_{n-v-1}(D_{n-v-2}(\ldots D_r(d)\ldots)))\ldots)\ .$$

where F is an arbitrary coefficient of the elements, except those of the first column, of each of the elements D_{n-v}. If we factor F out of each of the columns, except the first, of each of the elements D_{n-v}, which are themselves $(n-v)$-th roots of bordered determinants of order $n-v+2$, we obtain

$$D_{n-1}(\ldots D_{n-v+1}(F^{\frac{n-v+1}{n-v}}\ D_{n-v}(D_{n-v-1}(\ldots D_r(d)\ldots)))\ldots)\ .$$

Thus we see that we may advance F from the position of coefficient of D_{n-v-1} to the position of coefficient of D_{n-v} if we raise F to the power $(n-v+1)/(n-v)$. This is true for $v = 1, \ldots, r + 1$. Hence, to advance F to the position of coefficient of D_{n-1}, we must raise F to the power

$$\frac{n-v+1}{n-v} \cdot \frac{n-v+2}{n-v+1} \cdot \ \cdots\ \cdot \frac{n-1}{n-2} \cdot \frac{n}{n-1} = \frac{n}{n-v}\ .$$

Therefore we have the following lemma.

(2.5.10) <u>Lemma.</u> For an arbitrary coefficient F, we have

$$D_{n-1}(\ldots D_{n-v}(F\ D_{n-v-1}(D_{n-v-2}(\ldots D_r(d)\ldots)))\ldots)$$
$$= F^{n/(n-v)}\ D_{n-1}(\ldots D_r(d)\ldots)\ .$$

Now we set $A_{n-v} = (n-v+2)^{-1/(n-v)}$ and write (2.4.1) as

$$\left| 1 \quad x_j^i \right| = A_{n-1} D_{n-1}(d) \quad .$$

Clearly the elements $d_{j_1}^{i_1}$ of D_{n-1} are of the form $\left| 1 \quad x_j^i \right|$, with n replaced by $n-1$; so we apply (2.4.1) to them to obtain

$$\left| 1 \quad x_j^i \right| = A_{n-1} D_{n-1}(A_{n-2}(d)).$$

Applying (2.4.1), in turn, to the elements $d_{j_v}^{i_v}$ of each of the D_{n-v}, since they are of the form $\left| 1 \quad x_j^i \right|$, with n replaced by $n-v$, we complete the induction necessary to obtain finally

$$\left| 1 \quad x_j^i \right| = A_{n-1} D_{n-1}(A_{n-2} D_{n-2}(\ldots A_{n-s} D_{n-s}(d)\ldots)) \quad .$$

Now we use (2.5.10) to obtain

$$\left| 1 \quad x_j^i \right| = \prod_{v=1}^{s} A_{n-v}^{n/(n-v+1)} D_{n-1}(\ldots D_{n-s}(d)\ldots) \quad .$$

Finally we set $B_{n-s,n} = \prod_{v=1}^{s} A_{n-v}^{n/(n-v+1)}$ and replace s by $n-r$ to complete the proof of (2.5.9).

Chapter 3

k-Cells: Affine Subspaces

3.1. Definitions and Notation. Let K be a commutative field without characteristic and $L_n(K)$, $n = 1,2,\ldots$, be the set of all n-tuples of elements of K, $x = (x_1,\ldots,x_n)$; if $y = (y_1,\ldots,y_n)$ we write, by definition,

$$
(3.1.1)\qquad
\begin{aligned}
x + y &= (x_1 + y_1,\ldots,x_n + y_n),\\
ax &= (ax_1,\ldots,ax_n) \qquad\qquad (a \text{ in } K),\\
\theta &= (0,\ldots,0),\\
-x &= (-x_1,\ldots,-x_n).
\end{aligned}
$$

Then $L_n(K)$ is an n-dimensional vector space over K. We shall refer to $L_n(K)$ as the base space and call the elements x of $L_n(K)$ the vectors, or sometimes points, in the base space. When K is the field of real (or complex) numbers we shall simply denote the base space by R_n (or C_n). We shall assume at all times, unless the contrary is explicitly stated, that $L_n(K)$ is referred to the natural coordinate system or basis

$$
(3.1.2)\qquad
\begin{aligned}
e^1 &= (1,0,\ldots,0),\\
e^2 &= (0,1,\ldots,0),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
e^n &= (0,0,\ldots,1).
\end{aligned}
$$

We shall assume, further, a knowledge of the elementary theory of finite-dimensional vector spaces, see [2, chap. VII], [7], [8, chap. II], and [10].

Let $D_k(L_n(K))$, $k = 0,1,2,\ldots,$ be the set of all (k+1)-tuples of elements of $L_n(K)$, $((x^0,\ldots,x^k))$. We shall call the elements of $D_k(L_n(K))$ k-cells in $L_n(K)$, or simply k-cells.

If $((x^0,x^1,\ldots,x^k))$ be an arbitrary k-cell, we shall call the 0-cells $((x^i))$, the 1-cells $((x^0 , x^i)),\ldots,$ $((x^{i-1} , x^i))$, $((x^{i+1} , x^i)),\ldots,$ $((x^k , x^i))$, and the (k-1)-cell $((x^0,\ldots,x^{i-1}, x^{i+1},\ldots,x^k))$ respectively its vertices, its edges at the vertex $((x^i))$, and its face opposite the vertex $((x^i))$, $i = 0,1,\ldots,k$. In particular, $((x^0))$ will be called its initial vertex. The (k+1)xn matrix

$$(3.1.3)\qquad \begin{bmatrix} x_1^0 & \cdot & x_n^0 \\ \cdot & \cdot & \cdot \\ x_1^k & \cdot & x_n^k \end{bmatrix}$$

will be called the matrix of its vertices. The kxn matrix, $0 < k$,

$$(3.1.4)\qquad \begin{bmatrix} x_1^0 - x_1^i & \cdot & x_n^0 - x_n^i \\ \cdot & \cdot & \cdot \\ x_1^{i-1} - x_1^i & \cdot & x_n^{i-1} - x_n^i \\ x_1^{i+1} - x_1^i & \cdot & x_n^{i+1} - x_n^i \\ \cdot & \cdot & \cdot \\ x_1^k - x_1^i & \cdot & x_n^k - x_n^i \end{bmatrix}$$

will be called the matrix of its edges at the vertex $((x^i))$. If every matrix (3.1.4), $i = 0,\dots,k$, has rank k, we say the k-cell $((x^0,\dots,x^k))$ is a proper k-cell; otherwise, we call $((x^0,\dots,x^k))$ a null k-cell. For completeness, we define every 0-cell to be a proper 0-cell and say that the matrix of its edges has rank 0. Now, from the determinant identity

(3.1.5)

$$\begin{vmatrix} x^0_{j_1} - x^i_{j_1} & \cdot & x^0_{j_k} - x^i_{j_k} \\ \cdot & \cdot & \cdot \\ x^{i-1}_{j_1} - x^i_{j_1} & \cdot & x^{i-1}_{j_k} - x^i_{j_k} \\ x^{i+1}_{j_1} - x^i_{j_1} & \cdot & x^{i+1}_{j_k} - x^i_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} - x^i_{j_1} & \cdot & x^k_{j_k} - x^i_{j_k} \end{vmatrix} = (-)^i \begin{vmatrix} x^1_{j_1} - x^0_{j} & \cdot & x^1_{j_k} - x^0_{j_k} \\ \cdot & \cdot & \cdot \\ x^i_{j_1} - x^0_{j_1} & \cdot & x^i_{j_k} - x^0_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} - x^0_{j_1} & \cdot & x^k_{j_k} - x^0_{j_k} \end{vmatrix},$$

we see that the following theorem is true.

(3.1.6) <u>Theorem.</u> A k-cell is a proper k-cell if, and only if, the matrix of its edges at the initial vertex has rank k.

Let $((x^0,\dots,x^k))$ be a proper k-cell and suppose that (3.1.3), its matrix of vertices, has rank less than k. Then at least two row vectors are a linear combination of the remaining $k-1$ row vectors. Let these two vectors be x^{i_1} and x^{i_2}. We have

$$x^{i_1} = \sum_{\substack{\nu=0 \\ \nu \neq i_1, i_2}}^{k} a_\nu x^\nu, \qquad x^{i_2} = \sum_{\substack{\nu=0 \\ \nu \neq i_1, i_2}}^{k} b_\nu x^\nu.$$

Hence

$$x^{i_1} - x^{i_2} = \sum_{\substack{\nu=0 \\ \nu \neq i_1, i_2}}^{k} c_\nu x^\nu, \qquad c_\nu = a_\nu - b_\nu.$$

Now, taking $i = i_2$ in (3.1.4), we see that the matrix of edges at the initial vertex must have rank less than k. But this contradicts (3.1.5). Hence we have the following corollary.

(3.1.6) <u>Corollary.</u> The matrix of vertices of a proper k-cell never has rank less than k.

Since the rank of a matrix is never greater than the number of its columns, we have the following theorem.

(3.1.7) <u>Theorem.</u> Every k-cell in $L_n(K)$ for which $k > n$ is a null k-cell.

<u>3.2. Affine Subspaces.</u> Let $((x^0,\ldots,x^k))$ be an arbitrary k-cell in $L_n(K)$, and let X be the matrix of its edges at $((x^0))$, see (3.1.4). Clearly, the row vectors of X are elements of $L_n(K)$ and, as such, span a linear manifold M which is contained in $L_n(K)$. In fact, if M_i is the linear manifold spanned by the matrix of edges of $((x^0,\ldots,x^k))$ at $((x^i))$, $i = 1,2,\ldots,k$, then, since the sum and the difference of any two vectors in a linear manifold are also vectors in the manifold, it follows from

the identities $x^r - x^i = (x^r - x^0) - (x^i - x^0)$ and $x^r - x^0 = (x^r - x^i) - (x^0 - x^i)$ that $M_i = M$, $i = 1,2,\ldots,k$. We call the set S of all points y in $L_n(K)$ for which $y - x^0$ is in M the affine subspace of $L_n(K)$ determined by $((x^0,\ldots,x^k))$, and by the dimension of S we mean the dimension of M. It follows, therefore, that $((x^0,\ldots,x^k))$ determines a k-dimensional affine subspace S_k if, and only if, M is a k-dimensional linear manifold. Hence, in view of (3.1.6), we have the following corollary.

(3.2.1) Corollary. A k-cell determines a k-dimensional affine subspace if, and only if, it is a proper k-cell.

By definition, y is in the affine subspace S determined by $((x^0,\ldots,x^k))$ if, and only if, the vector $y - x^0$ is in the linear manifold M spanned by the row vectors of the matrix of the edges of $((x^0,\ldots,x^k))$ at the vertex $((x^0))$; that is, y is in S if, and only if, there are k elements $c_1,\ldots,c_k$ of K such that

$$y - x^0 = \sum_{i=1}^{k} c_i (x^i - x^0),$$

which can be expressed in the equivalent form

$$y = (1 - \sum_{i=1}^{k} c_i)\, x^0 + \sum_{i=1}^{k} c_i x^i.$$

Setting $c_0 = 1 - \sum_{i=1}^{k} c_i$, we have the following theorem.

(3.2.2) Theorem. A point y is in the affine subspace determined by the k-cell $((x^0,\ldots,x^k))$ if, and only if, there are $(k+1)$ elements $c_0,\ldots,c_k$ of the ground field K such that

$$y = \sum_{i=0}^{k} c_i x^i \quad \text{and} \quad \sum_{i=0}^{k} c_i = 1$$

hold simultaneously.

Taking $c_j = 1$ and $c_i = 0$ for $i \neq j$ we have $x^j = \sum_{i=0}^{k} c_i x^i$ and $\sum_{i=0}^{k} c_i = 1$ for $0 \leq j \leq k$. Hence the following corollary is true.

(3.2.3) Corollary. Every vertex of a k-cell is a point of the affine subspace which it determines.

Let $x^0,\ldots,x^k$, $0 < k \leq n$, be an arbitrary set of $(k+1)$ points of $L_n(K)$. The vectors $x^i - x^0$, $i = 1,\ldots,k$, span a linear manifold of dimension k if, and only if, there is no linear manifold of dimension $(k-1)$ containing the entire set. Hence we have the following theorem.

(3.2.4) Theorem. $(k+1)$ points of $L_n(K)$ determine a k-dimensional affine subspace if, and only if, they are contained in no $(k-1)$-dimensional affine subspace, $0 < k \leq n$.

Remark. Geometrically, a 0-dimensional affine subspace is a point, a 1-dimensional affine subspace is a line, a 2-dimensional affine subspace is a plane, etc., see [2, pp. 260-263].

(3.2.5) <u>Theorem.</u> A proper k-cell determines an affine subspace which contains the origin (the point θ) if, and only if, its matrix of vertices has rank k.

<u>Proof:</u> Let $((x^0,\ldots,x^k))$ be any proper k-cell and S_k the affine subspace which it determines. If θ is in S_k then, by (3.2.2),

$$\text{(i)} \qquad \theta = \sum_{i=0}^{k} c_i x^i \quad \text{and}$$

$$\text{(ii)} \qquad \sum_{i=0}^{k} c_i = 1$$

hold simultaneously. It follows from (ii) that not all c_i are zero. Hence, from (i), we see that the row vectors of the matrix of vertices of $((x^0,\ldots,x^k))$ has rank less than $(k+1)$. By (3.1.6), it never has rank less than k. Hence it has rank k. Conversely, if the matrix of vertices has rank k, its row vectors are linearly dependent, which means that there are $(k+1)$ elements $a_0,\ldots,a_k$, not all zero, of the ground field such that

$$\sum_{i=0}^{k} a_i x^i = \theta.$$

Now if it were true that $\sum_{i=0}^{k} a_i = 0$, then we would have

$$\theta = \left(\sum_{i=0}^{k} a_i\right) x^0 + \sum_{i=1}^{k} a_i(x^i - x^0)$$

$$= \sum_{i=1}^{k} a_i(x^i - x^0)$$

and, hence, by (3.1.6), since $((x^0,\ldots,x^k))$ is, by hypothesis, a proper k-cell,

$$a_1 = a_2 = \dots = a_k = 0.$$

But then $a_0 = a_1 = \dots = a_k = 0$, which is impossible. It follows therefore that

$$\sum_{i=0}^{k} a_i = a \neq 0.$$

Setting $c_i = a_i/a$, we have

$$\theta = \frac{1}{a}\,\theta = \frac{1}{a}\sum_{i=0}^{k} a_i x^i = \sum_{i=0}^{k} \frac{a_i}{a}\, x^i = \sum_{i=0}^{k} c_i x^i$$

as well as $\displaystyle\sum_{i=0}^{k} c_i = \sum_{i=0}^{k} \frac{a_i}{a} = \frac{1}{a}\sum_{i=0}^{k} a_i = \frac{1}{a}\, a = 1;$

hence, by (3.2.2), θ is in S_k. This completes the proof.

(3.2.6) <u>Theorem.</u> A necessary and sufficient condition that the affine subspaces determined by two proper k-cells in $I_n(K)$ should coincide is that the matrix of edges at the initial vertex of one be equal to that of the other pre-multiplied by a non-singular $k\times k$ matrix over K and that the affine subspaces determined by them have a point in common.

For let $((x^0,\dots,x^k))$ be a proper k-cell and S_k be the affine subspace determined by it. Then a necessary condition that any other proper k-cell $((y^0,\dots,y^k))$ lie in S_k or, what amounts to the same thing, determines the same affine subspace, is that each of the vectors $y^i - y^0$

be a linear combination of the vectors $x^i - x^0$, that is, that

$$y^i - y^0 = \sum_{j=1}^{k} c_j^i \,(x^j - x^0), \quad i = 1,\ldots,k.$$

In matrix notation, this becomes

$$\begin{bmatrix} y_1^1 - y_1^0 & \cdot & y_n^1 - y_n^0 \\ \cdot & \cdot & \cdot \\ y_1^k - y_1^0 & \cdot & y_n^k - y_n^0 \end{bmatrix} = \begin{bmatrix} c_1^1 & \cdot & c_k^1 \\ \cdot & \cdot & \cdot \\ c_1^k & \cdot & c_k^k \end{bmatrix} \begin{bmatrix} x_1^1 - x_1^0 & \cdot & x_n^1 - x_n^0 \\ \cdot & \cdot & \cdot \\ x_1^k - x_1^0 & \cdot & x_n^k - x_n^0 \end{bmatrix}.$$

Clearly, S_k has a point in common with itself. This proves the condition necessary. Conversely, if R_k is the affine subspace determined by the proper k-cell $((y^0,\ldots,y^k))$ and z is the point common to S_k and R_k, then, by definition, $z - x_0$ is a vector in the linear manifold M_x spanned by the row vectors of the matrix of edges at the initial vertex of $((x^0,\ldots,x^k))$, and $z - y^0$ is in the linear manifold M_y spanned by the row vectors of the matrix of edges at the initial vertex of $((y^0,\ldots,y^k))$. If, further,

(3.2.7) $$Y = C\,X,$$

where C is a nonsingular $k{\times}k$ matrix over K and X, Y are respectively the matrices of edges at the initial vertices of $((x^0,\ldots,x^k))$, $((y^0,\ldots,y^k))$ then

(3.2.8) $$X = C^{-1}Y.$$

Now let y be any point in R_k. Then, by definition, $y - y^0$ is in M_y, that is, there are elements $b_1, \ldots, b_k$ of K such that

$$\begin{aligned} y - y^0 &= \sum_{i=1}^{k} b_i (y^i - y^0) \\ &= \sum_{i=1}^{k} b_i \sum_{j=1}^{k} c^i_j (x^j - x^0) \qquad \text{by (3.2.7)} \\ &= \sum_{j=1}^{k} \Big(\sum_{i=1}^{k} b_i c^i_j \Big) (x^j - x^0) \quad , \end{aligned}$$

which shows that $y - y^0$ is in M_x. Similarly, we can show that $z - y^0$ is in M_x. We already know that $z - x^0$ is in M_x. Therefore, since

$$y - x^0 = (y - y^0) - (z - y^0) + (z - x^0),$$

we see that $y - x^0$ is also in M_x, which means that y is in S_k. On the other hand, if x is any point in S_k, then, by definition, $x - x^0$ is in M_x; and so there are elements $a_1, \ldots, a_k$ of K such that

$$\begin{aligned} x - x^0 &= \sum_{i=1}^{k} a_i (x^i - x^0) \\ &= \sum_{i=1}^{k} a_i \sum_{j=1}^{k} \left[C^{-1}\right]^i_j (y^j - y^0) \qquad \text{by (3.2.8)} \\ &= \sum_{v=1}^{k} \Big(\sum_{i=1}^{k} a_i \left[C^{-1}\right]^i_j \Big) (y^j - y^0), \end{aligned}$$

which shows that $x - x^0$ is in M_y. Similary, it can be shown that $z - x^0$ is in M_y. We already know that $z - y^0$ is in M_y. Consequently, we see, from

$$x - y^0 = (x - x^0) - (z - x^0) + (z - y^0),$$

that $x - y^0$ is also in M_y and, hence, by definition, x is in R_k. Thus, we have also proved the condition sufficient.

<u>3.3 k-Vectors and Grassmann Coordinates.</u> Let $((x^0, x^1))$ be an arbitrary 1-cell (directed line-segment) in R_3, as shown in Fig. 1. If p_1, p_2, p_3 are the measures of the projections of $((x^0, x^1))$ on the axes of x_1, x_2, x_3, we know that

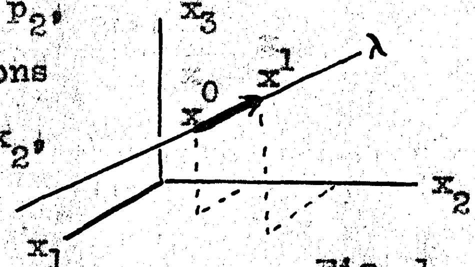

Fig. 1

$$p_1 = x_1^1 - x_1^0, \quad p_2 = x_2^1 - x_2^0, \quad p_3 = x_3^1 - x_3^0.$$

For purely heuristic reasons, we write these equations in the following determinant form.

(3.3.1)

$$p_1 = \frac{1}{1!}\begin{vmatrix} 1 & x_1^0 \\ 1 & x_1^1 \end{vmatrix}, \quad p_2 = \frac{1}{1!}\begin{vmatrix} 1 & x_2^0 \\ 1 & x_2^1 \end{vmatrix}, \quad p_3 = \frac{1}{1!}\begin{vmatrix} 1 & x_3^0 \\ 1 & x_3^1 \end{vmatrix}.$$

In terms of p_1, p_2, p_3, the length L of $((x^0, x^1))$ is

$$L = +\sqrt{p_1^2 + p_2^2 + p_3^2}.$$

If ϕ_1, ϕ_2, ϕ_3 are the angles (between 0 and π inclusive) which $((x^0, x^1))$ makes with the positive axes of x_1, x_2,

x_3, then

$$p_i = L \cos \phi_i , \quad i = 1, 2, 3.$$

From Analytic Geometry, we know that $\cos \phi_1$, $\cos \phi_2$, $\cos \phi_3$ are the direction cosines of the line λ on which $((x^0, x^1))$ lies, when λ is directed in the same sense as $((x^0, x^1))$. Hence p_1, p_2, p_3 are, themselves, direction components of the line λ. We shall call them the components of the 1-cell $((x^0, x^1))$.

It is evident geometrically that two directed line-segments (1-cells) which lie on the same line (S_1 in R_3), or on parallel lines, and have the same sense and same length, have the same components. Conversely, if p_1, p_2, p_3 are three numbers, not all zero, there are infinitely many directed line-segments (1-cells), in fact, one issuing from each point in space, which have the components p_1, p_2, p_3. Each pair of these have the same direction, the same sense, and the same length. If the components alone are known, the directed line-segment (1-cell) is free to move throughout space (R_3). The 1-cell is then called a vector. We shall sometimes call it a 1-vector. If its components are not all zero, a vector (1-vector) is called a proper vector (proper 1-vector); otherwise, it is called a null vector (null 1-vector).

More generally, let $((x^0, x^1, x^2))$ be an arbitrary 2-cell in R_3, as shown in Fig. 2. If p_{12}, p_{13}, p_{23} are the signed areas of the perpendicular projections of the oriented triangle (2-cell) $((x^0, x^1, x^2))$ on the x_1x_2, x_1x_3, x_2x_3 coordinate planes, we know that

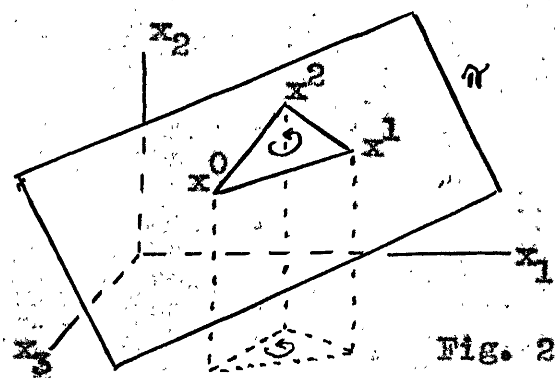

Fig. 2

(3.3.2)
$$p_{12} = \frac{1}{2!}\begin{vmatrix} 1 & x_1^0 & x_2^0 \\ 1 & x_1^1 & x_2^1 \\ 1 & x_1^2 & x_2^2 \end{vmatrix}, \quad p_{13} = \frac{1}{2!}\begin{vmatrix} 1 & x_1^0 & x_3^0 \\ 1 & x_1^1 & x_3^1 \\ 1 & x_1^2 & x_3^2 \end{vmatrix}, \quad p_{23} = \frac{1}{2!}\begin{vmatrix} 1 & x_2^0 & x_3^0 \\ 1 & x_2^1 & x_3^1 \\ 1 & x_2^2 & x_3^2 \end{vmatrix}.$$

In terms of p_{12}, p_{13}, p_{23}, the area A of $((x^0, x^1, x^2))$ is

$$A = +\sqrt{p_{12}^2 + p_{13}^2 + p_{23}^2}\,.$$

If ϕ_{12}, ϕ_{13}, ϕ_{23} are the angles which $((x^0, x^1, x^2))$ makes with the coordinate planes x_1x_2, x_1x_3, x_2x_3, then

$$p_{ij} = A\cos\phi_{ij}\,, \qquad ij = 12,\ 13,\ 23.$$

We call $\cos\phi_{12}$, $\cos\phi_{13}$, $\cos\phi_{23}$ the direction cosines of the plane π (the S_2 in R_3) on which $((x^0, x^1, x^2))$ lies, when the positive orientation on π has the same "sense" as $((x^0, x^1, x^2))$. Hence p_{12}, p_{13}, p_{23} are, themselves, direction components of the plane π. We shall call them the components of the 2-cell $((x^0, x^1, x^2))$.

It is evident geometrically that two 2-cells which lie on the same plane, or parallel planes, and have the same "sense" and same area, have the same components. Conversely, if p_{12}, p_{13}, p_{23} are three numbers, not all zero, there are infinitely many 2-cells which have the components p_{12}, p_{13}, p_{23}. Each pair of these have the same "direction", the same "sense", and the same area. If the components alone are known, the 2-cell will be called a 2-vector. If the components are not all zero it will be called a proper 2-vector; otherwise, it will be called a null 2-vector.

Let us generalize the ideas involved in the two illustrations just considered. To this end we make the following definitions.

(3.3.3) <u>Definition.</u> If $((x^0,\ldots,x^k))$ be an arbitrary k-cell in $L_n(K)$ then the $C(n,k)$ quantities

$$\frac{1}{k!}\begin{vmatrix} 1 & x^0_{j_1} & x^0_{j_2} & \cdot & x^0_{j_k} \\ 1 & x^1_{j_1} & x^1_{j_2} & \cdot & x^1_{j_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & x^k_{j_2} & \cdot & x^k_{j_k} \end{vmatrix}, \qquad 1 \leqq j_1 < j_2 < \cdots < j_k \leqq n,$$

are to be called the components of the k-cell $((x^0,\ldots,x^k))$.

Each of the determinants in (3.3.3) is of order $(k+1)$ with the first column composed entirely of 1's and with the remaining k columns selected from the matrix of vertices of $((x^0,\ldots,x^k))$.

(3.3.4) <u>Definition.</u> The $C(n,k)$-tuple

$$\begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^k \end{pmatrix} = \left(\ldots, \frac{1}{k!}\begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix}, \ldots\right)$$

of components of the k-cell $((x^0,\ldots,x^k))$ taken in lexicographical order will be called the k-vector determined by $((x^0,\ldots,x^k))$. It will be called a proper k-vector if its components are not all zero; otherwise, it will be called a null k-vector.

From 1.2, (3.1.4), and the identity

$$\begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} = \begin{vmatrix} x^1_{j_1} - x^0_{j_1} & \cdot & x^1_{j_k} - x^0_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} - x^0_{j_1} & \cdot & x^k_{j_k} - x^0_{j_k} \end{vmatrix}$$

we have the following corollary to (3.3.4).

(3.3.5) <u>Corollary.</u> If $((x^0,\ldots,x^k))$ is an arbitrary k-cell in $L_n(K)$ and X is the matrix of its edges at the initial vertex, then the row vector of the $1\times C(n,k)$ matrix $(1/k!)\, X^{(k)}$ is the k-vector determined by $((x^0,\ldots,x^k))$.

By (3.1.6), a k-cell is a proper k-cell if, and only if, the matrix of its edges at the initial vertex has rank k, which, in turn, is true, by (1.4.4) if, and only if, the k-th compound of the matrix of its edges has rank 1. Taking $0! = 1$ and recalling, (1.2.1), that the zero-th compound of a matrix is to be taken as the matrix containing the single element 1, we have, in all cases, in view of (3.3.4) and (3.3.5), the following theorem.

(3.3.6) <u>Theorem.</u> A k-cell is a proper k-cell if, and only if, it determines a proper k-vector.

In view of (3.2.1), we have the following corollary.

(3.3.7) <u>Corollary.</u> A k-cell in $L_n(K)$ determines a k-dimensional affine subspace of $L_n(K)$ if, and only if, it determines a proper k-vector.

By (3.2.6), we know that the affine subspaces S_k and R_k determined respectively by two proper k-cells $((x^0,\ldots,x^k))$ and $((y^0,\ldots,y^k))$ coincide if, and only if, they have a point in common and $Y = C\,X$, where C is a non-singular $k\times k$ matrix over the ground field and X, Y are respectively the matrices of edges at the initial vertices of $((x^0,\ldots,x^k))$, $((y^0,\ldots,y^k))$. Taking the k-th compound of both sides of this equation, we have

$$Y^{(k)} = |C|\,X^{(k)}.$$

Multiplying both members of this equation by $1/k!$ and recalling (3.3.5), we see that the following theorem is true.

(3.3.8) Theorem. The affine subspaces determined by two proper k-cells in $L_n(K)$ coincide if they have a point in common and the corresponding components of the k-vectors determined by the k-cells are proportional; and conversely.

If we remark that a linear manifold in $L_n(K)$ is an affine subspace which contains the origin, we are led to the following corollary to (3.3.8), in view of (3.2.5).

(3.3.9) Corollary. Two proper k-cells determine the same linear manifold in $L_n(K)$ if, and only if, their matrices of vertices have rank k and their corresponding components are proportional.

For completeness, we make the following definition.

(3.3.10) Definition. The affine subspaces determined by two proper k-cells with corresponding components proportional are said to be parallel.

Remark: When $L_n(K)$ is referred to homogeneous coordinates it is called a projective number space of dimension n over the ground field K, and the linear subspaces of $L_n(K)$ are the only subspaces which enter into consideration. The components of any proper k-cell which lies in a particular linear subspace then have the form

$$\begin{vmatrix} x^0_{j_0} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_0} & \cdot & x^k_{j_k} \end{vmatrix}$$

and are called the Grassmann coordinates of that linear subspace. For example, see [8, chap. VII].

Let $((x^0,\ldots,x^k))$ be an arbitrary k-cell, and let T be a translation which carries each point x in $L_n(K)$ into another point $y = x + t$ (t is a point in $L_n(K)$). Then T carries $((x^0,\ldots,x^k))$ into the k-cell $((y^0,\ldots,y^k)) = ((Tx^0,\ldots,Tx^k))$. Moreover, we have, for the k-vectors determined by these k-cells,

$$\begin{aligned}
\begin{pmatrix} y^0 \\ \vdots \\ y^k \end{pmatrix} &= \begin{pmatrix} Tx^0 \\ \vdots \\ Tx^k \end{pmatrix} \\
&= \begin{pmatrix} x^0+t \\ \vdots \\ x^k+t \end{pmatrix} \\
&= \left(\ldots,\ 1/k! \begin{vmatrix} 1 & x^0_{j_1}+t_{j_1} & \cdot & x^0_{j_k}+t_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1}+t_{j_1} & \cdot & x^k_{j_k}+t_{j_k} \end{vmatrix},\ \ldots\right) \\
&= \left(\ldots,\ 1/k! \begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix},\ \ldots\right) \\
&= \begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} .
\end{aligned}$$

Hence we have the following theorem.

(3.3.11) Theorem. The components of a k-cell are left unchanged by a translation.

The next corollary follows immediately.

(3.3.12) Corollary. A translation carries proper k-cells into proper k-cells.

From (3.3.12), since an affine subspace is completely determined by a proper k-cell, we have another corollary.

(3.3.13) Corollary. A translation carries the affine subspace determined by a proper k-cell into the affine subspace determined by the image k-cell.

From the determinant identity

$$\begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^i_{j_1} & \cdot & x^i_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^j_{j_1} & \cdot & x^j_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} = - \begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^j_{j_1} & \cdot & x^j_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^i_{j_1} & \cdot & x^i_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix},$$

follows the next theorem.

(3.3.14) Theorem. The interchange of two vertices has the effect of changing the sign of each of the components of a k-cell.

The following more general theorem is an immediate consequence of (3.3.14).

(3.3.15) <u>Theorem.</u> Let $((x^0,\ldots,x^k))$ be an arbitrary k-cell, and let $((x^{i_0},\ldots,x^{i_k}))$ be the k-cell obtained from it by the permutation $(i_0\ldots i_k)$ of its vertices. Then the components of $((x^{i_0},\ldots,x^{i_k}))$ are those, or the negatives of those, of $((x^0,\ldots,x^k))$ according as $(i_0\ldots i_k)$ is an even, or an odd, permutation.

<u>3.4. Addition of k-Vectors and Multiplication by Elements of the Ground Field.</u>

If $$\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} = (\ldots,\ p_{j_1\ldots j_k},\ \ldots)$$ and

$$\begin{pmatrix} y^0 \\ \vdots \\ y^k \end{pmatrix} = (\ldots,\ q_{j_1\ldots j_k},\ \ldots)$$

are respectively the k-vectors determined by the k-cells $((x^0,\ldots,x^k))$ and $((y^0,\ldots,y^k))$, we write, <u>by definition,</u>

$$(3.4.1)\quad \begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} + \begin{pmatrix} y^0 \\ \vdots \\ y^k \end{pmatrix} = (\ldots,\ p_{j_1\ldots j_k} + q_{j_1\ldots j_k},\ \ldots),$$

$$(3.4.2)\qquad a\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} = (\ldots,\ ap_{j_1\ldots j_k},\ \ldots) \qquad (a \text{ in } K),$$

$$(3.4.3)\qquad 0\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} = (\ldots,\ 0,\ \ldots)\ ,$$

$$(3.4.4)\qquad -\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} = (\ldots,\ -p_{j_1\ldots j_k},\ \ldots)\ .$$

We have no assurance that the so defined operations on k-vectors lead to k-vectors. However, we shall see that they do in certain instances, and these are the cases in which we are primarily interested at the present. For example, we can express (3.3.11), (3.3.14), and (3.3.15) as follows:

$$(3.4.5)\qquad \begin{pmatrix} x^0+t \\ \vdots \\ x^k+t \end{pmatrix} = \begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix}\ ,$$

$$(3.4.6)\qquad \begin{pmatrix} x^0 \\ \vdots \\ x^i \\ \vdots \\ x^j \\ \vdots \\ x^k \end{pmatrix} = -\begin{pmatrix} x^0 \\ \vdots \\ x^j \\ \vdots \\ x^i \\ \vdots \\ x^k \end{pmatrix}\ , \qquad 0 \leq i \neq j \leq k,$$

$$(3.4.7)\qquad \begin{pmatrix} x^{i_0} \\ \vdots \\ x^{i_k} \end{pmatrix} = \delta^{i_0\ldots i_k}\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix}\ ,$$

where $\delta^{i_0\ldots i_k} = 1$ or -1 according as $(i_0\ldots i_k)$ is an even or odd permutation of the integers $0, 1, \ldots, k$.

Now we can prove the following theorem.

(3.4.8) Theorem. The components of a k-cell are respectively equal to the sums of the corresponding components of the k+1 consistently oriented k-cells which the original k-cell determines with respect to the origin.

Proof: $$\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} = \left(\ldots, 1/k! \begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix}, \ldots \right)$$

$$= \left(\ldots, 1/k! \sum_{i=0}^{k} \theta^i \begin{vmatrix} x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot \\ x^{i-1}_{j_1} & \cdot & x^{i-1}_{j_k} \\ x^{i+1}_{j_1} & \cdot & x^{i+1}_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix}, \ldots \right)$$

$$= \sum_{i=0}^{k} \left(\ldots, 1/k! \begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^{i-1}_{j_1} & \cdot & x^{i-1}_{j_k} \\ 1 & 0 & \cdot & 0 \\ 1 & x^{i+1}_{j_1} & \cdot & x^{i+1}_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix}, \ldots \right)$$

$$= \sum_{i=0}^{k} \begin{pmatrix} x^0 \\ \vdots \\ x^{i-1} \\ \theta \\ x^{i+1} \\ \vdots \\ x^k \end{pmatrix} .$$

Applying (3.4.5), we obtain the following corollary.

(3.4.9) <u>Corollary.</u> $\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} = \sum_{i=0}^{k} \begin{pmatrix} x^0 \\ \vdots \\ x^{i-1} \\ y \\ x^{i+1} \\ \vdots \\ x^k \end{pmatrix}$,

where y is an arbitrary point of $L_n(K)$.

(3.4.10) <u>Remark.</u> The set of 1-cells with the origin as initial vertex is isomorphic with the set $L_n(K)$ of vectors which constitutes the base space. By (3.4.8), we can express the components of the directed line-segment joining any point x^0 with any other point x^1 as the difference of the corresponding components of two 1-cells with the origin as initial vertex; for we have

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} \theta \\ x^1 \end{pmatrix} + \begin{pmatrix} x^0 \\ \theta \end{pmatrix}$$

$$= \begin{pmatrix} \theta \\ x^1 \end{pmatrix} - \begin{pmatrix} \theta \\ x^0 \end{pmatrix}.$$

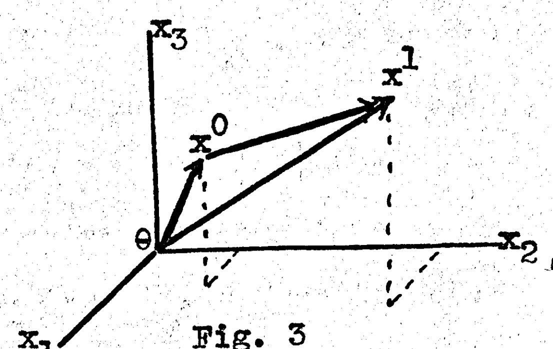

Fig. 3

This is just ordinary vector addition, as shown in Fig. 3.

Using (3.4.1) and the identity

$$\begin{vmatrix} 1 & 0 & \cdot & 0 \\ 1 & x^1_{j_1} & \cdot & x^1_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^{i-1}_{j_1} & \cdot & x^{i-1}_{j_k} \\ 1 & x^i_{j_1}+y_{j_1} & \cdot & x^i_{j_k}+y_{j_k} \\ 1 & x^{i+1}_{j_1} & \cdot & x^{i+1}_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} = \begin{vmatrix} x^1_{j_1} & \cdot & x^1_{j_k} \\ \cdot & \cdot & \cdot \\ x^{i-1}_{j_1} & \cdot & x^{i-1}_{j_k} \\ x^i_{j_1}+y_{j_1} & \cdot & x^i_{j_k}+y_{j_k} \\ x^{i+1}_{j_1} & \cdot & x^{i+1}_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix}$$

$$= \begin{vmatrix} x^1_{j_1} & \cdot & x^1_{j_k} \\ \cdot & \cdot & \cdot \\ x^i_{j_1} & \cdot & x^i_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} + \begin{vmatrix} x^1_{j_1} & \cdot & x^1_{j_k} \\ \cdot & \cdot & \cdot \\ y_{j_1} & \cdot & y_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \cdot & 0 \\ 1 & x^1_{j_1} & \cdot & x^1_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^i_{j_1} & \cdot & x^i_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} + \begin{vmatrix} 1 & 0 & \cdot & 0 \\ 1 & x^1_{j_1} & \cdot & x^1_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & y_{j_1} & \cdot & y_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix},$$

we obtain the next theorem.

(3.4.11) Theorem. For k-vectors determined by k-cells with initial vertex at the origin, we have

$$\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^{i-1} \\ x^i + y \\ x^{i+1} \\ \vdots \\ x^k \end{pmatrix} = \begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^{i-1} \\ x^i \\ x^{i+1} \\ \vdots \\ x^k \end{pmatrix} + \begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^{i-1} \\ y \\ x^{i+1} \\ \vdots \\ x^k \end{pmatrix} .$$

Similarly, we obtain the next theorem.

(3.4.12) Theorem. For k-vectors determined by k-cells with initial vertex at the origin, we have

$$\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^{i-1} \\ ax^i \\ x^{i+1} \\ \vdots \\ x^k \end{pmatrix} = a \begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^{i-1} \\ x^i \\ x^{i+1} \\ \vdots \\ x^k \end{pmatrix} .$$

3.5. Representations of the Components of a k-Cell.

Let X^j_{kn} denote the j-th component of the k-cell $((x^0, \ldots, x^k))$. Then

$$X^j_{kn} = (1/k!) \begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} .$$

Now if we apply (2.5.9) to the determinant in the right member, we have, for $0 < r < k$,

$$\begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} = B_{rk} D_{k-1}(D_{k-2}(\ldots D_r(d)\ldots)).$$

Recalling that each of the elements $D_r(d)$ is of the form

$$\begin{vmatrix} 1 & d^i_j \end{vmatrix}^{1/r} ,$$

we factor $r!$ from each of the columns, except the first, of the elements $D_r(d)$, and then apply (2.5.10) to obtain

$$X^j_{kn} = (1/k!)\ (r!)^{k/(k-r)} B_{rk} D_{k-1}(\ldots D_r(d/r!)\ldots) ,$$

where $B_{rk} = \prod_{v=1}^{k-k} \left(\frac{1}{k-v+2}\right)^{\frac{k}{(k-v)(k-v+1)}}$.

For the sake of brevity we denote

$$(1/k!)\ (r!)^{k/(k-r)} B_{rk} D_{k-1}(\ldots D_r(d/r!)\ldots)$$

by $E_r(X^j_{kn})$. Then we have

$$(3.5.1) \quad \begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} = \left(E_r(X^1_{kn}), \ldots, E_r(X^{C(n,k)}_{kn})\right),$$

which expresses the components of $((x^0,\dots,x^k))$ in terms of the components of the r-dimensional faces of the faces of the faces ... of the faces of $((x^0,\dots,x^k))$ whenever the base space is algebraically closed.

We can obtain another evaluation of the components of $((x^0,\dots,x^k))$ in terms of the components of its r-cells by using (2.3.8). To this end, let $|A|$, in (2.3.8), be replaced by $k!\, X^j_{kn}$; then in (2.3.8), take $h = 1$, $k = r$, and let $A^1_1 = 1$ be the upper left corner element of $k!\, X^j_{kn}$. We have, as a result, replaced n, in (2.3.8), by $k + 1$; hence, if we replace B by B^j_{kn} accordingly, we get

$$\left| B^j_{kr} \right| = (k!\, X^j_{kn})^{C(k-1,r-1)}, \qquad 0 < r \leqq k.$$

Then, if the base space is algebraically closed, we can put this in the form

$$X^j_{kn} = (1/k!)\,(r!)^{k/r} \left| B^j_{kr}(1/r!) \right|^{1/C(k-1,r-1)},$$

where

$$\left[B^j_{kr}\right]^u_v = \begin{vmatrix} 1 & x^0_{j_{v_1}} & \cdot & x^0_{j_{v_r}} \\ 1 & x^{i_{u_1}}_{j_{v_1}} & \cdot & x^{i_{u_1}}_{j_{v_r}} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^{i_{u_r}}_{j_{v_1}} & \cdot & x^{i_{u_r}}_{j_{v_r}} \end{vmatrix}$$

and u,v run over $1, 2, \dots, C(k,r)$ as $(u_1,\dots,u_r)$,

$(v_1,\ldots,v_r)$ run over the r-tuples of $1, 2, \ldots, k$ in lexicographical order. Again for the sake of brevity, we denote

$$(1/k!)\,(r!)^{k/r}\left|B^j_{kr}(1/r!)\right|^{1/C(k-1,r-1)}$$

by $F_r(X^j_{kn})$ to obtain

$$(3.5.2)\quad \begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} = \left(F_r(X^1_{kn}),\ \ldots,\ F_r(X^{C(n,k)}_{kn})\right),$$

which expresses the components of $((x^0,\ldots,x^k))$ in terms of the components of its r-cells having a common initial vertex.

We can obviously obtain many other representations for the components of $((x^0,\ldots,x^k))$ by using (2.3.8) in other ways.

Chapter 4

The Generalized Inner Product, Norm, and Distance

Throughout this chapter we shall assume the base space to be C_n, except when otherwise indicated.

4.1. Definitions and Notation. If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are elements in C_n and a is a complex number, we write, by definition,

$$(4.1.1) \qquad (x, y) = \sum_{i=1}^{n} x_i \bar{y}_i \quad , \qquad \text{and}$$

$$(4.1.2) \qquad \|x\| = (x, x)^{\frac{1}{2}} \quad ,$$

where the super-imposed bar denotes the complex conjugate. The bilinear form (4.1.1) is called the inner product of the vector x with the vector y and possesses the usual properties, namely,

$$(4.1.3) \qquad (y, x) = \overline{(x, y)} \quad ,$$

$$(4.1.4) \qquad (x^0 + x^1, y) = (x^0, y) + (x^1, y) \quad ,$$

$$(4.1.5) \qquad (ax, y) = a(x, y) \quad ,$$

$$(4.1.6) \qquad (x, x) \geqq 0; \quad (x, x) = 0 \text{ if, and only if, } x = \theta.$$

The non-negative, real number $\|x\|$ is called the norm of the vector x and has the usual properties of a norm, namely,

(4.1.7) $\|x\| \geqq 0$; $\|x\| = 0$ if, and only if, $x = \theta$,

(4.1.8) $\|ax\| = |a| \cdot \|x\|$,

where $|a|$ denotes the absolute value of the complex number a. Moreover, Schwarz's inequality

(4.1.9) $\left|(x, y)\right|^{\frac{1}{2}} = \|x\| \|y\|$

is also true. We define the distance of x and y to be

(4.1.10) $\|x, y\| = \|x - y\|$;

it possesses the requisite properties, namely,

(4.1.11) $\|x, y\| = \|y, x\|$

(4.1.12) $\|x, y\| \geqq 0$, $\|x, y\| = 0$ if, and only if, $x = y$.

(4.1.13) $\|x, y\| + \|y, z\| \geqq \|x, z\|$ (Triangle Inequality).

When the base space is R_n, the foregoing definitions still hold; however, property (4.1.3) then becomes merely

(4.1.14) $(y, x) = (x, y)$.

Now, by analogy, if $((x^0,\ldots,x^k))$, $((y^0,\ldots,y^k))$ are two k-cells in either C_n or R_n, we write, by definition.

(4.1.15)

$$\begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \Bigg| \begin{bmatrix} y^0 \\ \vdots \\ y^k \end{bmatrix} = (1/k!)^2 \sum_{j=1}^{C(n,k)} \begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} \begin{vmatrix} 1 & y^0_{j_1} & \cdot & y^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & y^k_{j_1} & \cdot & y^k_{j_k} \end{vmatrix}^*$$

where the summation extends over the $C(n,k)$ components of the k-cells, and where the asterisk denotes the complex conjugate transposed determinant, and

(4.1.16)
$$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\| = \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \Bigg| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right]^{\frac{1}{2}} .$$

We call (4.1.15) and (4.1.16) respectively the inner product of the k-cell $((x^0,\ldots,x^k))$ with the k-cell $((y^0,\ldots,y^k))$, the norm of the k-cell $((x^0,\ldots,x^k))$. We define the distance of $((x^0,\ldots,x^k))$ and $((y^0,\ldots,y^k))$ to be

(4.1.17)
$$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \Bigg| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\| = \left\| \begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} - \begin{pmatrix} y^0 \\ \vdots \\ y^k \end{pmatrix} \right\| ,$$

where the meaning of $\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix} - \begin{pmatrix} y^0 \\ \vdots \\ y^k \end{pmatrix}$ is clear in view of (3.4.1) and (3.4.4).

4.2. Properties of the Generalized Inner Product.

When the base space is C_n, the inner product has the following properties.

(4.2.1) $$\left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right] = \overline{\left[\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\,\middle|\,\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\right]} .$$

(4.2.2) $$\left[\left(\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\right) + \left(\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right)\,\middle|\,\begin{matrix}z^0\\ \vdots\\ z^k\end{matrix}\right] = \left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}z^0\\ \vdots\\ z^k\end{matrix}\right] + \left[\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\,\middle|\,\begin{matrix}z^0\\ \vdots\\ z^k\end{matrix}\right] .$$

(4.2.3) $$\left[a\left(\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\right)\,\middle|\,\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right] = a\left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right]$$ (a is a complex number)

(4.2.4) $$\left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\right] \geqq 0; \quad \left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\right] = 0 \text{ if, and only if,}$$

there exist k complex numbers $a_0,\ldots,a_{j-1}, a_{j+1},\ldots,a_k$ such that both $\sum_{\substack{i=0\\ i\neq j}}^{k} a_i = 1$ and $x^j = \sum_{\substack{i=0\\ i\neq j}}^{k} a_i x^i$ for some j.

(4.2.5) $$\left[\begin{matrix}x^0\\ \vdots\\ x^i\\ \vdots\\ x^j\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \cdot\\ \\ \cdot\\ \\ \cdot\\ y^k\end{matrix}\right] = -\left[\begin{matrix}x^0\\ \vdots\\ x^j\\ \vdots\\ x^i\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \cdot\\ \\ \cdot\\ \\ \cdot\\ y^k\end{matrix}\right] .$$

(4.2.6) $$\left[\begin{matrix}\theta\\ ax^1\\ x^2\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \cdot\\ \cdot\\ \vdots\\ y^k\end{matrix}\right] = a\left[\begin{matrix}\theta\\ x^1\\ x^2\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \cdot\\ \cdot\\ \vdots\\ y^k\end{matrix}\right] .$$

(4.2.7) $$\left[\begin{matrix}x^0 + z\\ \vdots\\ x^k + z\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right] = \left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right] .$$

(4.2.8) $$\left[\begin{matrix} ax^0 \\ \vdots \\ ax^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = a^k \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] .$$

(4.2.9) $$\left|\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right]\right| \leqq \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\right]^{\frac{1}{2}} \left[\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right]^{\frac{1}{2}}$$ (Schwarz' Inequality)

(4.2.10) $$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\left(\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right) + \left(\begin{matrix} z^0 \\ \vdots \\ z^k \end{matrix}\right)\right] = \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] + \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} z^0 \\ \vdots \\ z^k \end{matrix}\right] .$$

(4.2.11) $$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\, a\left(\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right)\right] = \bar{a} \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] .$$

(4.2.12) $$\left[\begin{matrix} x^{i_0} \\ \vdots \\ x^{i_k} \end{matrix}\,\middle|\,\begin{matrix} y^{j_0} \\ \vdots \\ y^{j_k} \end{matrix}\right] = \delta^{i_0\cdots i_k}\delta^{j_0\cdots j_k} \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] ,$$

where $\delta^{i_0\cdots i_k}$ (or $\delta^{j_0\cdots j_k}$) $= +1$ or -1 according as $(i_0\cdots i_k)$ (or $(j_0\cdots j_k)$) is an even or an odd permutation of the integers $0,1,\ldots,k$.

(4.2.13) $$\left[\begin{matrix} \theta_1 \\ x^1 \\ \vdots \\ ax^i \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \cdot \\ \cdot \\ \cdot \\ y^k \end{matrix}\right] = a \left[\begin{matrix} \theta_1 \\ x^1 \\ \vdots \\ x^i \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \cdot \\ \cdot \\ \cdot \\ y^k \end{matrix}\right] ; \quad \left[\begin{matrix} x^0 \\ \cdot \\ \cdot \\ \cdot \\ x^k \end{matrix}\,\middle|\,\begin{matrix} \theta_1 \\ y^1 \\ \vdots \\ by^j \\ \vdots \\ y^k \end{matrix}\right] = \bar{b} \left[\begin{matrix} x^0 \\ \cdot \\ \cdot \\ \cdot \\ x^k \end{matrix}\,\middle|\,\begin{matrix} \theta_1 \\ y^1 \\ \vdots \\ y^j \\ \vdots \\ y^k \end{matrix}\right] .$$

(4.2.14) $$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 + z \\ \vdots \\ y^k + z \end{matrix}\right] = \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] .$$

(4.2.15) $$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} ay^0 \\ \vdots \\ ay^k \end{matrix}\right] = (\bar{a})^k \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] .$$

If the base space is R_n, then all of the above properties still hold. However (4.2.1) then becomes simply

(4.2.16)
$$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = \left[\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\right]$$

and similarly for (4.2.11) and (4.2.12).

For convenience of notation, let

$$\begin{pmatrix} \theta_1 \\ x^1 \\ \vdots \\ z^i \\ \vdots \\ x^k \end{pmatrix}$$

denote the k-vector determined by the k-cell

$$((\theta,\ x^1,\ \ldots,\ x^{i-1},\ z^i,\ x^{i+1},\ \ldots,\ x^k)).$$

Using (3.4.11), (3.4.12), (4.2.1), (4.2.2), and (4.2.3), we obtain the following theorem.

(4.2.17) <u>Theorem.</u> We have, for k-cells with initial vertex at the origin,

$$\left[\begin{matrix} \theta_1 \\ x^1 \\ \vdots \\ a_i x^i + c_i z^i \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} \theta_1 \\ y^1 \\ \vdots \\ b_j y^j + c_j z^j \\ \vdots \\ y^k \end{matrix}\right] = a_i \bar{b}_j \left[\begin{matrix} \theta_1 \\ x^1 \\ \vdots \\ x^i \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} \theta_1 \\ y^1 \\ \vdots \\ y^j \\ \vdots \\ y^k \end{matrix}\right] + a_i \bar{c}_j \left[\begin{matrix} \theta_1 \\ x^1 \\ \vdots \\ x^i \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} \theta_1 \\ y^1 \\ \vdots \\ z^j \\ \vdots \\ y^k \end{matrix}\right]$$
$$+ c_i \bar{b}_j \left[\begin{matrix} \theta_1 \\ x^1 \\ \vdots \\ z^i \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} \theta_1 \\ y^1 \\ \vdots \\ y^j \\ \vdots \\ y^k \end{matrix}\right] + c_i \bar{c}_j \left[\begin{matrix} \theta_1 \\ x^1 \\ \vdots \\ z^i \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} \theta_1 \\ y^1 \\ \vdots \\ z^j \\ \vdots \\ y^k \end{matrix}\right].$$

The above properties do not constitute an independent set of properties; in fact, the last eight are easily deduced from the first seven properties. (4.2.1), (4.2.2), and (4.2.3) follow immediately from (3.3.4) and (4.1.15). (4.2.4) follows directly from (4.1.15). One obtains (4.2.5) from (3.4.6) and (4.2.3), (4.2.6) from (3.1.1), (3.4.12), and (4.2.3), and (4.2.7) from (3.4.5).

4.3. Properties of the Generalized Norm and Distance.

Now it is easy to see that the generalized norm possesses the following properties.

(4.3.1) $$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\| \geqq 0 \; ; \quad \left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\| = 0$$

if, and only if, there exist k complex numbers $a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k$ such that both

$$\sum_{\substack{i=0 \\ i \neq j}}^{k} a_i = 1 \quad \text{and} \quad x^j = \sum_{\substack{i=0 \\ i \neq j}}^{k} a_i x^i$$

hold for some integer j, $0 \leqq j \leqq k$.

(4.3.2) $$\left\| \begin{matrix} ax^0 \\ \vdots \\ ax^k \end{matrix} \right\| = |a|^k \left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\| \quad .$$

(4.3.3) $$\left\| \begin{matrix} x^{i_0} \\ \vdots \\ x^{i_k} \end{matrix} \right\| = \left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\| \quad .$$

for every permutation $(i_0 \dots i_k)$ of the integers $0, 1, \dots, k$. While the generalized distance has the following properties.

(4.3.4) $$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\| = \left\| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\| .$$

(4.3.5) $$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\| \geqq 0 \; ; \; \left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\| = 0$$

if, and only if, $((x^0, \dots, x^k))$ and $((y^0, \dots, y^k))$ have the same components.

(4.3.6) $$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\| + \left\| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \middle| \begin{matrix} z^0 \\ \vdots \\ z^k \end{matrix} \right\| \geqq \left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} z^0 \\ \vdots \\ z^k \end{matrix} \right\|$$ (Triangle Inequality).

In view of (4.1.16), it is clear that (4.3.1), (4.3.2), (4.3.3) respectively follow from (4.2.4), (4.2.8) and (4.2.15), (4.2.12).

In view of (4.2.2) and (4.2.10), we have the following corollary to (3.4.9).

(4.3.7) <u>Corollary.</u>

$$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right] = \sum_{i,j=0}^{k} \left[\begin{matrix} x^0 \\ \vdots \\ x^{i-1} \\ u \\ x^{i+1} \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^{j-1} \\ v \\ y^{j+1} \\ \vdots \\ y^k \end{matrix} \right] .$$

Now, using (4.1.16), we obtain a generalization of the cosine law for vectors. We state it as a theorem.

(4.3.8) <u>Theorem.</u>

$$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\|^2 = \sum_{i=0}^{k} \left\| \begin{matrix} x^0 \\ \vdots \\ x^{i-1} \\ y \\ x^{i+1} \\ \vdots \\ x^k \end{matrix} \right\|^2 + \sum_{\substack{i,j=0 \\ i \neq j}}^{k} \left[\begin{matrix} x^0 \\ \vdots \\ x^{i-1} \\ y \\ x^{i+1} \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^{j-1} \\ y \\ x^{j+1} \\ \vdots \\ x^k \end{matrix} \right] .$$

When the base space is R_n, we have, using (4.1.16) and (4.1.17),

$$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\|^2 = \left[\left(\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right) - \left(\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right) \middle| \left(\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right) - \left(\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right) \right]$$

Then, by (4.2.1), (4.2.2), (4.2.10), and (4.1.16), we have

$$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\|^2 = \left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\|^2 - 2 \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right] + \left\| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\|^2 .$$

Thus we have another generalization of the cosine law for two vectors. We state it as a theorem, also.

(4.3.9) <u>Theorem.</u> When the base space is R_n,

$$\left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\|^2 = \left\| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right\|^2 + \left\| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right\|^2 - 2 \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right] .$$

Next we state (4.3.1) in a different manner as a theorem.

(4.3.10) <u>Theorem.</u> The norm of a k-cell in C_n is non-negative and vanishes when, and only when, it is a null k-cell.

This follows from (4.1.15) taken along with (3.3.4) and (3.3.6).

4.4. Representations of the Generalized Inner Product. Besides the form (4.1.15) in which the generalized inner product has been defined, it is possible to express it in several other ways, as in the following. If we diminish the elements of each of the succeeding rows of the determinants in (4.1.15) by the corresponding elements of the first row, we obtain

(4.4.1) $$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = (1/k!)^2 \text{ times}$$

$$\sum_{j=1}^{C(n,k)} \begin{vmatrix} x^1_{j_1} - x^0_{j_1} & \cdot & x^1_{j_k} - x^0_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} - x^0_{j_1} & \cdot & x^k_{j_k} - x^0_{j_k} \end{vmatrix} \begin{vmatrix} y^1_{j_1} - y^0_{j_1} & \cdot & y^1_{j_k} - y^0_{j_k} \\ \cdot & \cdot & \cdot \\ y^k_{j_1} - y^0_{j_1} & \cdot & y^k_{j_k} - y^0_{j_k} \end{vmatrix}^* .$$

If we designate the matrix of edges at the initial vertex of $((x^0,\ldots,x^k))$ by X and that of $((y^0,\ldots,y^k))$ by Y, see (3.1.4), then, by 1.2 and (1.1.4), we have the representation

(4.4.2) $$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = (1/k!)^2 \left| X^{(k)} (Y^{(k)})^* \right| .$$

In view of the Binet-Cauchy multiplication theorem for determinants, see [11, pp. 77-78], we deduce the next representation from (4.4.2).

$$(4.4.3)\qquad \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \begin{vmatrix} y^0 \\ \vdots \\ y^k \end{vmatrix} = (1/k!)^2 \left| X\, Y^* \right| .$$

The element in the i-th row and j-th column of the determinant in (4.4.3) is

$$\sum_{v=1}^{n} (x_v^i - x_v^0)\, \overline{(y_v^j - y_v^0)} .$$

Hence, in view of (4.1.1), we have, for $i,j = 1,\dots,k$,

$$(4.4.4)\qquad \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \begin{vmatrix} y^0 \\ \vdots \\ y^k \end{vmatrix} = (1/k!)^2 \left| (x^i - x^0,\ y^j - y^0) \right| ,$$

which expresses the inner product of $((x^0,\dots,x^k))$ with $((y^0,\dots,y^k))$ in terms of the ordinary inner products of their edges at their initial vertices.

Now, noting that

$$\sum_{v=1}^{n} (x_v^i - x_v^0)\overline{(y_v^j - y_v^0)} = \sum_{v=1}^{n} \begin{vmatrix} 1 & x_v^0 \\ 1 & x_v^i \end{vmatrix} \begin{vmatrix} 1 & y_v^0 \\ 1 & y_v^j \end{vmatrix}^* ,$$

we have, using (4.1.15), the representation

$$(4.4.5)\qquad \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \begin{vmatrix} y^0 \\ \vdots \\ y^k \end{vmatrix} = (1/k!)^2 \left| \begin{bmatrix} x^0 \\ x^i \end{bmatrix} \begin{vmatrix} y^0 \\ y^j \end{vmatrix} \right| , \qquad i,j = 1,\dots,k,$$

which expresses the inner product of two k-cells in terms of the inner products of the 1-cells at their initial vertices.

From (4.4.4), we have, by (4.1.3) and (4.1.4),

$$\left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right] = (1/k!)^2 \left| (x^i,y^j)-(x^0,y^j)-(x^i,y^0)+(x^0,y^0) \right|$$

$$= \frac{-1}{(k!)^2}\begin{vmatrix} 0 & 0 & \cdot & 0 & 1 \\ 0 & (x^1,y^1)-(x^0,y^1)-(x^1,y^0)+(x^0,y^0) & \cdot & (x^1,y^k)-(x^0,y^k)-(x^1,y^0)+(x^0,y^0) & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & (x^k,y^1)-(x^0,y^1)-(x^k,y^0)+(x^0,y^0) & \cdot & (x^k,y^k)-(x^0,y^k)-(x^k,y^0)+(x^0,y^0) & 1 \\ 1 & 1 & \cdot & 1 & 0 \end{vmatrix}.$$

Now if each element of the i-th row be augmented by $(x^i,y^0)-(x^0,y^0)$ times the corresponding element of the last row and each element of the j-th column be augmented by (x^0,y^j) times the corresponding element of the last column, $i,j = 0,1,\ldots,k$, there results the representation

$$(4.4.6)\qquad \left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right] = -\,(1/k!)^2 \begin{vmatrix} (x^i,y^j) & 1 \\ 1 & 0 \end{vmatrix} ,$$

which expresses the inner product as a bordered determinant--see the notation employed in 2.4--whose entries are ordinary inner products of vectors of the base space.

By the same procedure, starting with (4.4.5), we obtain the representation

$$(4.4.7)\ \left[\begin{matrix}x^0\\ \vdots\\ x^k\end{matrix}\,\middle|\,\begin{matrix}y^0\\ \vdots\\ y^k\end{matrix}\right] = -\,(1/k!)^2 \begin{vmatrix} \left[\begin{matrix}\theta\\ x^i\end{matrix}\,\middle|\,\begin{matrix}\theta\\ y^j\end{matrix}\right] & 1 \\ 1 & 0 \end{vmatrix},\quad i,j = 0,1,\ldots,k,$$

which is, in effect, just a restatement of (4.4.6).

Taking the r-th compound of the determinant on the right in (4.4.4), we obtain, by (1.4.9), for $i,j = 1,\dots,k$,

$$(k!)^2 \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{C(k-1,r-1)} = \left| \left[(x^i - x^0,\ y^j - y^0) \right]^{(r)} \right|$$

$$= \left| \begin{array}{ccc} \begin{vmatrix} (x^1-x^0, y^1-y^0) & \cdots & (x^1-x^0, y^r-y^0) \\ \vdots & \vdots & \vdots \\ (x^r-x^0, y^1-y^0) & \cdots & (x^r-x^0, y^r-y^0) \end{vmatrix} & \cdots & \begin{vmatrix} (x^1-x^0, y^{k-r+1}-y^0) & \cdots & (x^1-x^0, y^k-y^0) \\ \vdots & \vdots & \vdots \\ (x^r-x^0, y^{k-r+1}-y^0) & \cdots & (x^r-x^0, y^k-y^0) \end{vmatrix} \\ \vdots & \vdots & \vdots \\ \begin{vmatrix} (x^{k-r+1}-x^0, y^1-y^0) & \cdots & (x^{k-r+1}-x^0, y^r-y^0) \\ \vdots & \vdots & \vdots \\ (x^k-x^0, y^1-y^0) & \cdots & (x^k-x^0, y^r-y^0) \end{vmatrix} & \cdots & \begin{vmatrix} (x^{k-r+1}-x^0, y^{k-r+1}-y^0) & \cdots & (x^{k-r+1}-x^0, y^k-y^0) \\ \vdots & \vdots & \vdots \\ (x^k-x^0, y^{k-r+1}-y^0) & \cdots & (x^k-x^0, y^k-y^0) \end{vmatrix} \end{array} \right|$$

$$= (r!)^{2C(k,r)} \left| \begin{array}{ccc} \begin{bmatrix} x^0 & y^0 \\ x^1 & y^1 \\ \vdots & \vdots \\ x^r & y^r \end{bmatrix} & \cdots & \begin{bmatrix} x^0 & y^0 \\ x^1 & y^{k-r+1} \\ \vdots & \vdots \\ x^r & y^k \end{bmatrix} \\ \vdots & \vdots & \vdots \\ \begin{bmatrix} x^0 & y^0 \\ x^{k-r+1} & y^1 \\ \vdots & \vdots \\ x^k & y^r \end{bmatrix} & \cdots & \begin{bmatrix} x^0 & y^0 \\ x^{k-r+1} & y^{k-r+1} \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix} \end{array} \right| .$$

Hence we have the following generalization of (4.4.5):

$$(4.4.8)\qquad \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{C(k-1,r-1)} = \frac{(r!)^{2C(k,r)}}{(k!)^{2C(k-1,r-1)}} \left| d^i_j \right| ,$$

where $d^i_j = \begin{bmatrix} x^0 & y^0 \\ x^{i_1} & y^{j_1} \\ \vdots & \vdots \\ x^{i_r} & y^{j_r} \end{bmatrix}$, $i,j = 1,\ldots,C(k,r)$, and the order is lexicographical order. Now it is clear that whenever the base space is algebraically closed (4.4.8) expresses the inner product of the k-cells $((x^0,\ldots,x^k))$ and $((y^0,\ldots,y^k))$ in terms of the inner products of the r-cells of $((x^0,\ldots,x^k))$ having the same initial vertex as $((x^0,\ldots,x^k))$ with those of $((y^0,\ldots,y^k))$ having a common initial vertex with $((y^0,\ldots,y^k))$.

Let $((x^0,\ldots,x^n))$ be an arbitrary n-cell in the base space. Then, recalling (2.4.0) and (3.3.4), we can write

$$(4.4.9)\qquad n! \begin{pmatrix} x^0 \\ \vdots \\ x^n \end{pmatrix} = \left(\left| 1 \quad x^i_j \right| \right),$$

$$(4.4.10)\qquad (n-1)! \begin{pmatrix} x^0 \\ \vdots \\ x^{n-i-1} \\ x^{n-i+1} \\ \vdots \\ x^n \end{pmatrix} = (\hat{x}^i_1, \ldots, \hat{x}^i_n), \quad \text{and}$$

$$(4.4.11) \qquad (n-1)! \begin{pmatrix} x^0 \\ \vdots \\ x^{n-i-1} \\ x^{n-i+1} \\ \vdots \\ x^n \end{pmatrix} = (\ldots,(-1)^{i+j}\, \check{x}^i_j, \ldots) \quad ,$$

where $i = 0,1,\ldots,n$ and $j = 1,2,\ldots,n$. Now $\hat{x}^i = (\hat{x}^i_1, \ldots, \hat{x}^i_n)$ and $\check{x}^i = (\check{x}^i_1, \ldots, \check{x}^i_n)$ are clearly elements in $L_n(K)$. Hence, with each n-cell $((x^0, \ldots, x^n))$ in $L_n(K)$, we can associate the n-cells $((\hat{x}^0, \ldots, \hat{x}^n))$ and $((\check{x}^0, \ldots, \check{x}^n))$. Then, as in (4.4.9), we have

$$(4.4.12) \qquad n! \begin{pmatrix} \hat{x}^0 \\ \vdots \\ \hat{x}^n \end{pmatrix} = \left(\left\| 1 \quad \hat{x}^i_j \right\| \right) \quad \text{and}$$

$$(4.4.13) \qquad n! \begin{pmatrix} \check{x}^0 \\ \vdots \\ \check{x}^n \end{pmatrix} = \left(\left\| 1 \quad \check{x}^i_j \right\| \right) \quad .$$

Applying (4.1.15) to (4.4.12) and (4.4.13), we obtain respectively

$$(n!)^2 \left[\begin{matrix} \hat{x}^0 \\ \vdots \\ \hat{x}^n \end{matrix} \,\middle|\, \begin{matrix} \hat{y}^0 \\ \vdots \\ \hat{y}^n \end{matrix} \right] = \left\| 1 \quad \hat{x}^i_j \right\| \left\| 1 \quad \hat{y}^i_j \right\|^*$$

$$= \left(\left\| 1 \quad x^i_j \right\| \left\| 1 \quad y^i_j \right\|^* \right)^{n-1} \quad \text{or} \quad 0 \qquad \text{by (2.4.2),}$$

according as n is an even or an odd integer, and

$$(n!)^2 \left[\begin{matrix} \check{x}^0 \\ \vdots \\ \check{x}^n \end{matrix} \,\middle|\, \begin{matrix} \check{y}^0 \\ \vdots \\ \check{y}^n \end{matrix} \right] = \left\| 1 \quad \check{x}^i_j \right\| \left\| 1 \quad \check{y}^i_j \right\|^*$$

Therefore we have, by (2.4.1),

$$(n!)^2 \left[\begin{matrix}\check{x}^0\\ \vdots\\ \check{x}^n\end{matrix}\middle|\begin{matrix}\check{y}^0\\ \vdots\\ \check{y}^n\end{matrix}\right] = (n+1)^2 \left(\left\| 1 \quad x^i_j \right\| \left\| 1 \quad y^i_j \right\|^* \right)^{n-1} .$$

Using (4.1.15) and (4.4.9), we next deduce the relations

$$(4.4.14) \qquad \left[\begin{matrix}\hat{x}^0\\ \vdots\\ \hat{x}^n\end{matrix}\middle|\begin{matrix}\hat{y}^0\\ \vdots\\ \hat{y}^n\end{matrix}\right] = (n!)^{2n-4} \left[\begin{matrix}x^0\\ \vdots\\ x^n\end{matrix}\middle|\begin{matrix}y^0\\ \vdots\\ y^n\end{matrix}\right]^{n-1} \quad \text{or} \quad 0$$

according as n is an even or an odd integer, and

$$(4.4.15) \qquad \left[\begin{matrix}\check{x}^0\\ \vdots\\ \check{x}^n\end{matrix}\middle|\begin{matrix}\check{y}^0\\ \vdots\\ \check{y}^n\end{matrix}\right] = (n+1)^2 (n!)^{2n-4} \left[\begin{matrix}x^0\\ \vdots\\ x^n\end{matrix}\middle|\begin{matrix}y^0\\ \vdots\\ y^n\end{matrix}\right]^{n-1} .$$

When the base space is algebraically closed, these relations furnish further representations of the generalized inner product of two k-cells.

For convenience, let us write

$$(4.4.16) \qquad c^i_j = \left[\begin{matrix}x^0\\ \vdots\\ x^{n-i-1}\\ x^{n-i+1}\\ \vdots\\ x^n\end{matrix}\middle|\begin{matrix}y^0\\ \vdots\\ y^{n-j-1}\\ y^{n-j+1}\\ \vdots\\ y^n\end{matrix}\right] .$$

Then, applying (4.1.1) to (2.4.0), we have

$$(\check{x}^i, \check{y}^j) = \sum_{\nu=1}^{n} \check{x}^i_\nu \, \check{y}^i_\nu$$

$$= (-)^{i+j} \sum_{\nu=1}^{n} ((-)^{i+\nu} \check{x}^i_\nu) \overline{((-)^{j+\nu} \check{y}^j_\nu)}$$

Therefore we have, by (4.1.15) and (4.4.11),

$$(\check{x}^i, \check{y}^j) = (-)^{i+j}((n-1)!)^2 c^i_j \quad .$$

Using this in (4.4.6), we obtain

$$\left[\begin{matrix}\check{x}^0\\ \vdots \\ \check{x}^n\end{matrix} \middle| \begin{matrix}\check{y}^0\\ \vdots \\ \check{y}^n\end{matrix}\right] = -(1/n!)^2 \begin{vmatrix} (-)^{i+j}((n-1)!)^2 c^i_j & 1 \\ 1 & 0 \end{vmatrix} \quad .$$

Factoring $((n-1)!)^2$ out of each of the first $n+1$ rows and $1/((n-1)!)^2$ out of the last column yields

$$(4.4.17) \quad \left[\begin{matrix}\check{x}^0\\ \vdots \\ \check{x}^n\end{matrix} \middle| \begin{matrix}\check{y}^0\\ \vdots \\ \check{y}^n\end{matrix}\right] = -\frac{((n-1)!)^{2n}}{(n!)^2} \begin{vmatrix} (-)^{i+j} c^i_j & 1 \\ 1 & 0 \end{vmatrix} \quad .$$

Similarly, we obtain

$$(4.4.18) \quad \left[\begin{matrix}\hat{x}^0\\ \vdots \\ \hat{x}^n\end{matrix} \middle| \begin{matrix}\hat{y}^0\\ \vdots \\ \hat{y}^n\end{matrix}\right] = -\frac{((n-1)!)^{2n}}{(n!)^2} \begin{vmatrix} c^i_j & 1 \\ 1 & 0 \end{vmatrix} \quad .$$

Now, using (4.4.17) in (4.4.15) and (4.4.18) in (4.4.14), we obtain two further representations, namely,

$$(4.4.19) \quad \left[\begin{matrix}x^0\\ \vdots \\ x^n\end{matrix} \middle| \begin{matrix}y^0\\ \vdots \\ y^n\end{matrix}\right]^{n-1} = -\frac{(n!)^2}{n^{2n}(n+1)^2} \begin{vmatrix} (-)^{i+j} c^i_j & 1 \\ 1 & 0 \end{vmatrix} \quad ,$$

and, if n is an even integer,

$$(4.4.20)\quad \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^n & y^n \end{bmatrix}^{n-1} = -\frac{(n!)^2}{n^{2n}} \begin{vmatrix} c^i_j & 1 \\ 1 & 0 \end{vmatrix} ,$$

where c^i_j is defined in (4.4.16) and $i,j = 0,1,\ldots,n.$

We know by (4.4.6) that

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{k-1} = (-(k!)^{-2})^{k-1} \begin{vmatrix} (x^i,y^j) & 1 \\ 1 & 0 \end{vmatrix}^{k-1} \qquad i,j = 0,\ldots,k.$$

Using (2.4.4), we obtain next

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{k-1} = \frac{(-(k!)^{-2})^{k-1}}{(k+1)^2} \begin{vmatrix} \widetilde{(x^i,y^j)} & 1 \\ 1 & 0 \end{vmatrix} ,$$

where $\widetilde{(x^i,y^j)}$ is the cofactor of (x^i,y^j). Then, in view of (4.4.6),

$$\widetilde{(x^i,y^j)} = (-)^{i+j}(-((k-1)!)^2) \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^{i-1} & y^{j-1} \\ x^{i+1} & y^{j+1} \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix} .$$

Substituting this in and factoring $-((k-1)!)^2$ out of each of the first $k+1$ rows and $-((k-1)!)^{-2}$ out of the last column of

$$\begin{vmatrix} \widetilde{(x^i,y^j)} & 1 \\ 1 & 0 \end{vmatrix} ,$$

we obtain, after an obvious simplification,

$$(4.4.21)\qquad \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix}\!\!\left|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right]^{k-1} = -\frac{(k!)^2}{k^{2k}(k+1)^2}\begin{vmatrix} (-)^{i+j}b^i_j & 1 \\ 1 & 0 \end{vmatrix},$$

$$\text{where}\qquad b^i_j = \left[\begin{matrix} x^0 \\ \vdots \\ x^{i-1} \\ x^{i+1} \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^{j-1} \\ y^{j+1} \\ \vdots \\ y^k \end{matrix}\right], \qquad i,j = 0, 1, \ldots, k.$$

In particular, if we permute the first $k+1$ rows so that their order is reversed and do the same for the first $k+1$ columns, we have

$$(4.4.22)\qquad \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix}\!\!\left|\,\begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right]^{k-1} = -\frac{(k!)^2}{k^{2k}(k+1)^2}\begin{vmatrix} (-)^{i+j}c^i_j & 1 \\ 1 & 0 \end{vmatrix},$$

$$\text{where}\qquad c^i_j = \left[\begin{matrix} x^0 \\ \vdots \\ x^{k-i-1} \\ x^{k-i+1} \\ \vdots \\ x^k \end{matrix}\,\middle|\,\begin{matrix} y^0 \\ \vdots \\ y^{k-j-1} \\ y^{k-j+1} \\ \vdots \\ y^k \end{matrix}\right], \qquad i,j = 0, 1, \ldots, k.$$

Taking $k = n$, we see that (4.4.19) is merely a particular case of (4.4.22).

Now we shall show that (4.4.21) and (4.4.22) are implied by (4.4.8). It is clear that we can derive (4.4.21) from (4.4.22) by reversing the procedure employed in deriving (4.4.22) from (4.4.21). Hence we need only show that (4.4.22) can be derived from (4.4.8). Taking $r = k-1$ in (4.4.8), we have

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{k-1} = \frac{((k-1)!)^{2k}}{(k!)^{2k-2}} \left| d^i_j \right|, \qquad d^i_j = \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^{k-i} & y^{k-j} \\ x^{k-i+2} & y^{k-j+2} \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix},$$

$i,j = 1,\ldots,k$. Now, however, $\left| d^i_j \right| = \left| (-)^{i+j} d^i_j \right|$ and

$$(-)^{i+j} d^i_j = \begin{bmatrix} x^1 & y^1 \\ \vdots & \vdots \\ x^{k-i} & y^{k-j} \\ x^0 & y^0 \\ x^{k-i+2} & y^{k-j+2} \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}.$$

Therefore, if we put $a^i_j = (-)^{i+j} d^i_j$, we have

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{k-1} = (k!)^2 k^{-2k} \left| a^i_j \right|, \qquad i,j = 1,\ldots,k.$$

But

$$\begin{vmatrix} a^1_1 & \cdot & a^1_k \\ \cdot & \cdot & \cdot \\ a^k_1 & \cdot & a^k_k \end{vmatrix} = -(k+1)^{-2} \begin{vmatrix} a^1_1 & \cdot & a^1_k & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a^k_1 & \cdot & a^k_k & 0 & 1 \\ 0 & \cdot & 0 & 0 & k+1 \\ 1 & \cdot & 1 & k+1 & 0 \end{vmatrix}.$$

Hence

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{k-1} = -\frac{(k!)^2}{k^{2k}(k+1)^2} \begin{vmatrix} a_1^1 & . & a_k^1 & -\sum_{j=1}^{k} a_j^1 & 1 \\ . & . & . & . & . \\ a_1^k & . & a_k^k & -\sum_{j=1}^{k} a_j^k & 1 \\ -\sum_{i=1}^{k} a_1^i & . & -\sum_{i=1}^{k} a_k^i & \sum_{i,j=1}^{k} a_j^i & 1 \\ 1 & . & 1 & 1 & 0 \end{vmatrix}.$$

Taking $n = k$ in (4.4.16), we have

$$a_{j+1}^{i+1} = (-)^{i+j}\, c_j^i\,, \qquad i,j = 0, 1, \ldots, k-1,$$

and, in view of (4.3.7),

$$-\sum_{j=1}^{k} a_j^{i+1} = -\begin{bmatrix} x^1 & y^1 \\ \vdots & . \\ x^{k-i-2} & . \\ x^0 & . \\ x^{k-i} & . \\ \vdots & \\ x^k & y^k \end{bmatrix} = (-)^{i+k}\, c_k^i, \quad i = 0, 1, \ldots, k-1.$$

Similarly,

$$-\sum_{i=1}^{k} a_{j+1}^i = (-)^{k+j}\, c_j^k\,, \qquad j = 0, 1, \ldots, k-1;$$

$$-\sum_{i,j=1}^{k} a_j^i = (-)^{k+k}\, c_k^k \quad .$$

Therefore

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{k-1} = -\frac{(k!)^2}{k^{2k}(k+1)^2} \begin{vmatrix} (-)^{i+j}\, c_j^i & 1 \\ 1 & 0 \end{vmatrix},$$

where $i,j = 0,1,\ldots,k$, which is (4.4.22).

We can use a method similar to either of the two methods used to prove (4.4.22) to prove the next theorem.

(4.4.23) <u>Theorem.</u> According as k is an even or an odd integer, it is true that

$$-\frac{(k!)^2}{k^{2k}}\begin{vmatrix} c^i_j & 1 \\ 1 & 0 \end{vmatrix} = \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{k-1} \quad \text{or} \quad 0,$$

$$\text{where} \quad c^i_j = \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^{k-i-1} & y^{k-j-1} \\ x^{k-i+1} & y^{k-j+1} \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}, \qquad i,j = 0, \ldots, k.$$

It is easy to see that (4.4.20) is just a particular case of (4.4.23).

When the base space is algebraically closed (4.4.22) and (4.4.23) give two more representations for the inner product of two k-cells. Now it is natural to ask if we can find an expression which serves as a generalization of (4.4.6) in the same way in which (4.4.8) is a generalization of (4.4.5). It follows from (2.5.8) that we can not. However, we can obtain a further generalization of (4.4.5). To this end, let i,j respectively run over the sequence $1, 2, \ldots, C(k-h, r-h)$ as $(i_1,\ldots,i_{r-h})$, $(j_1,\ldots,j_{r-h})$ run over the sequence of the $(r-h)$-tuples, taken in lexicographical order, of the integers $h+1, h+2, \ldots, k$; let, further,

$$b_j^i = \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^h & y^h \\ x^{i_1} & y^{j_1} \\ \vdots & \vdots \\ x^{i_{r-h}} & y^{j_{r-h}} \end{bmatrix} \quad \text{and} \quad a_j^i = \begin{Vmatrix} (x^{u_i}, y^{v_j}) & 1 \\ 1 & 0 \end{Vmatrix},$$

where $u_i = 0, 1, \ldots, h, i_1, \ldots, i_{r-h}$ and $v_j = 0, 1, \ldots, h, j_1, \ldots, j_{r-h}$. Then it follows from (4.4.6) that

$$\left\| a_j^i \right\| = (-(r!)^2)^R \left\| b_j^i \right\|, \qquad R = C(k-h, r-h).$$

Applying (2.3.8) to the determinant on the left, we get

$$\begin{Vmatrix} (x^\alpha, y^\beta) & 1 \\ 1 & 0 \end{Vmatrix}^P \begin{Vmatrix} (x^\gamma, y^\delta) & 1 \\ 1 & 0 \end{Vmatrix}^Q = (-(r!)^2)^R \left\| b_j^i \right\|,$$

where $\alpha, \beta = 0, 1, \ldots, k$; $\gamma, \delta = 0, 1, \ldots, h$;

$$P = C(k-h-1, r-h-1); \qquad Q = C(k-h-1, r-h).$$

Using (4.4.6) again, we have

$$(-(k!)^2)^P (-(h!)^2)^Q \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^P \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^h & y^h \end{bmatrix}^Q = (-(r!)^2)^R \left\| b_j^i \right\|.$$

Noting that $P + Q = R$, we have the following theorem.

(4.4.24) Theorem.

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^P = \frac{(r!)^{2(P+Q)}}{(k!)^{2P}(h!)^{2Q}} \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^h & y^h \end{bmatrix}^{-Q} \left| b^i_j \right| ,$$

where

$$b^i_j = \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^h & y^h \\ x^{i_1} & y^{j_1} \\ \vdots & \vdots \\ x^{i_{r-h}} & y^{j_{r-h}} \end{bmatrix} ,$$

$P = C(k-h-1,\ r-h-1)$, $Q = C(k-h-1,\ r-h)$, $i,j = 1,2,\dots,$ $C(k-h,\ r-h)$, and the order is lexicographical order.

Now it is clear that whenever the base space is algebraically closed (4.4.24) expresses the inner product of the k-cells $((x^0,\dots,x^k))$ and $((y^0,\dots,y^k))$ in terms of (i) the inner products of the r-cells of $((x^0,\dots,x^k))$ having the same initial h-cell $((x^0,\dots,x^h))$ with those of $((y^0,\dots,y^k))$ having a common h-cell $((y^0,\dots,y^h))$, and (ii) the inner product of $((x^0,\dots,x^h))$ and $((y^0,\dots,y^h))$. Now let us take $\left[x^0 \mid y^0 \right] = 1$, in accordance with (4.4.6), and $0! = 1$; then we see that (4.4.8) follows from (4.4.24) as a corollary.

It should be remarked that slightly more general representations than (4.4.8) and (4.4.24) can be obtained by taking an arbitrary vertex and an arbitrary h-cell respectively instead of initial vertex and the initial h-cell.

Next we prove a theorem which expresses the inner product of two k-cells in C_n in terms of the inner products of the r-dimensional faces of the (r+1)-dimensional faces of the ... of the (k-1)-dimensional faces; namely,

(4.4.25) Theorem. When the base space is C_n,

$$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = B_r\, D_{k-1}\,(D_{k-2}(\ldots D_r(d)\ldots)),$$

where, $0 < r < k$,

$$B_r = (-1)^{(k-r)/r} \frac{(r!)^{2k/r}}{(k!)^2} \prod_{v=1}^{k-r} \left(\frac{1}{k-v+2}\right)^{\frac{2k}{(k-v)(k-v+1)}},$$

$$D_{k-v} = \begin{vmatrix} (-)^{i_v+j_v} d_{j_v}^{i_v} & 1 \\ 1 & 0 \end{vmatrix}^{\frac{1}{k-v}},$$

$$d_{j_v}^{i_v} = \left[\begin{matrix} x^{(i_1)(i_2)\cdots(i_v)}{}_0 \\ \vdots \\ x^{(i_1)(i_2)\cdots(i_v)}{}_{k-v} \end{matrix} \middle| \begin{matrix} y^{(j_1)(j_2)\cdots(j_v)}{}_0 \\ \vdots \\ y^{(j_1)(j_2)\cdots(j_v)}{}_{k-v} \end{matrix}\right],$$

$((i_1)_0, \ldots, (i_1)_{k-1})$ runs over the sequence of the k-tuples of the integers $0,1,\ldots,k$, and i_v, $1 < v < k$, runs over the sequence $0,1,\ldots,k-v+1$ as

$$\left((i_1)_{\cdot_{\cdot_{(i_v)_0}}}, \cdot \cdot \cdot \cdot, (i_1)_{\cdot_{\cdot_{(i_v)_{k-v}}}} \right)$$

runs over the sequence of the $(k-v+1)$-tuples of the integers

$$(i_1)_{\cdot_{\cdot_{(i_{v-1})_0}}}, \cdot \cdot \cdot \cdot, (i_1)_{\cdot_{\cdot_{(i_{v-1})_{k-v+1}}}},$$

the order being lexicographical order in all cases, and similarly for j_v.

In order to facilitate the proof, we take

$$A_{k-v} = \left(\frac{-((k-v+1)!)^2}{(k-v+1)^{2(k-v+1)}(k-v+2)^2} \right)^{\frac{1}{k-v}}, \qquad 0 < v < k.$$

Then, using the notation of (4.4.25), we have, by (4.4.22)

$$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right] = A_{k-1} \, D_{k-1} \, (d).$$

Now we, in turn, apply (4.4.22) to the elements $d_{j_1}^{i_1}$ of D_{k-1}, which are clearly inner products of $(k-1)$-cells; we deduce

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix} = A_{k-1}\ D_{k-1}\ (A_{k-2}\ D_{k-2}\ (d)).$$

Applying (4.4.22) to the elements $d_{j_v}^{i_v}$ of each of the factors D_{k-v}, since they are inner products of (k-v)-cells, we complete the induction which leads us to the conclusion:

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix} = A_{k-1}\ D_{k-1}\ (A_{k-2}\cdots D_{k-s+1}(A_{k-s}D_{k-s}(d))\cdots).$$

Here we have the inner product represented by an s term sequence of nested multiples of roots of bordered determinants. We can simplify this representation somewhat by factoring out the coefficients A_{k-v}, $v = 2, 3, \ldots, s$. To this end let us consider

$$D_{k-1}(D_{k-2}(\cdots D_{k-v+1}(FD_{k-v-1}(\cdots D_{k-s}(d)\cdots)))\cdots)),$$

where F is an arbitrary coefficient of the elements, except those of the last row and those of the last column, of each of the elements D_{k-v+1} which are themselves (k-v+1)-th roots of bordered determinants of order $k-v+4$. If we factor F out of each of the first $k-v+3$ rows and $1/F$ out of the last column of each of the elements D_{k-v+1}, we obtain

$$D_{k-1}(\cdots D_{k-v+2}(F^{\frac{k-v+2}{k-v+1}}\ D_{k-v+1}(D_{k-v}\cdots D_{k-s}(d)\cdots))\cdots).$$

Thus we see that advancing F from the position of coefficient of D_{k-v} to that of coefficient of D_{k-v+1} has the effect of raising F to the power $(k-v+2)/(k-v+1)$. Since v is any integer from 2 to s inclusive, it is clear that the effect of advancing F from the position of coefficient of D_{k-v} to that of coefficient of D_{k-1} is that of raising F to the power

$$\frac{k-v+2}{k-v+1}\ \frac{k-v+3}{k-v+2}\ \cdots\ \frac{k-1}{k-2}\ \frac{k}{k-1} = \frac{k}{k-v+1} \quad .$$

Therefore, we must conclude that

$$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\ \middle|\ \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = \prod_{\nu=1}^{s} A_{k-v}^{\frac{k}{k-v+1}}\ D_{k-1}\,(D_{k-2}\cdots D_{k-s}(d)\cdots).$$

Now, since

$$\prod_{\nu=1}^{s}(-1)^{\frac{k}{(k-v)(k-v+1)}} = (-1)^{k\sum_{\nu=1}^{s}\left(\frac{1}{k-v}-\frac{1}{k-v+1}\right)} = (-1)^{\frac{s}{k-s}} \quad ,$$

we have

$$\prod_{\nu=1}^{s} A_{k-v}^{\frac{k}{k-v+1}} = \prod_{\nu=1}^{s}\left(\frac{-((k-v+1)!)^2}{(k-v+1)^{2(k-v+1)}(k-v+2)^2}\right)^{\frac{k}{(k-v)(k-v+1)}}$$

$$= (-1)^{\frac{s}{k-s}} \left(\prod_{\nu=1}^{s} \left(\frac{(k-v)!}{(k-v+1)^{k-v}(k-v+2)} \right)^{\frac{1}{k-v} - \frac{1}{k-v+1}} \right)^{2k}$$

$$= (-1)^{\frac{s}{k-s}} \left(\frac{((k-s)!)^{\frac{1}{k-s}}}{(k!)^{1/k}} \prod_{\nu=1}^{s} \frac{(k-v+2)^{\frac{1}{k-v+1}}}{(k-v+2)^{\frac{1}{k-v}}} \right)^{2k}$$

$$= (-1)^{\frac{s}{k-s}} \frac{((k-s)!)^{\frac{2k}{k-s}}}{(k!)^2} \prod_{\nu=1}^{s} \left(\frac{1}{k-v+2} \right)^{\frac{2k}{(k-v)(k-v+1)}}$$

Setting $r = k - s$ and $B_r = \prod_{\nu=1}^{s} A_{k-v}^{\frac{k}{k-v+1}}$ completes the proof of (4.4.25).

Now if, in (2.2.3), we set $n = k$, $x = 1$, $S_0 = 1$, and $a_j^i = (x^i - x^0, y^j - y^0)$, we obtain

$$\left| \left[(x^i - x^0, y^j - y^0) \right] + I_k \right| = \sum_{r=0}^{k} S_r, \qquad i, j = 1, 2, \ldots, k,$$

where I_k is the unit matrix of order k and where, in view of (4.4.4),

$$S_r = \sum_{\nu=1}^{C(k,r)} T_v, \quad T_v = (r!)^2 \begin{bmatrix} x^{v_0} & y^{v_0} \\ \vdots & \vdots \\ x^{v_r} & y^{v_r} \end{bmatrix},$$

$v_0 = 0$, and v runs over $1, \ldots, C(k,r)$ as $(v_1, \ldots, v_r)$

runs over the r-tuples of $1, \ldots, k$ in lexicographical order. Now, if we take $\left[x^0 \middle| y^0\right] = 1$ and $0! = 1$, we have the following interesting relation.

$$(4.4.26) \qquad \left| \left[(x^i - x^0, y^j - y^0)\right] + I_k \right| = \sum_{r=0}^{k} \sum_{v=1}^{C(k,r)} T_v,$$

where
$$T_v = (r!)^2 \left[\begin{matrix} x^{v_0} \\ \vdots \\ x^{v_r} \end{matrix} \middle| \begin{matrix} y^{v_0} \\ \vdots \\ y^{v_r} \end{matrix}\right] \quad \text{and} \quad v_0 = 0.$$

We obtain two further representations from (3.5.1) and (3.5.2) respectively; namely,

$$(4.4.27) \qquad \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = \sum_{j=1}^{C(n,k)} E_r(X^j_{kn}) \, \overline{E_r(Y^j_{kn})}$$

and

$$(4.4.28) \qquad \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = \sum_{j=1}^{C(n,k)} F_r(X^j_{kn}) \, \overline{F_r(Y^j_{kn})} \, ,$$

where the superimposed bar denotes the complex conjugate.

When the base space is R_n, we have, by (4.1.2), (4.1.3), and (4.1.4),

$$(x^i, y^j) = \tfrac{1}{2} \left\{ \|x^i\|^2 - \|x^i - y^j\|^2 + \|y^j\|^2 \right\} .$$

Substituting this in (4.4.6), we get

$$\begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \Bigg| \begin{bmatrix} y^0 \\ \vdots \\ y^k \end{bmatrix} = -(k!)^{-2} \begin{vmatrix} \frac{1}{2}\left\{\|x^i\|^2 - \|x^i - y^j\|^2 + \|y^j\|^2\right\} & 1 \\ 1 & 0 \end{vmatrix} .$$

Now, if, in the determinant on the right, we diminish the elements of the i-th row by $\frac{1}{2}\|x^i\|^2$ times the corresponding elements of the last row, then decrease the elements of the j-th column by $\frac{1}{2}\|y^j\|^2$ times the corresponding elements of the last column, then factor $-\frac{1}{2}$ out of each row except the last, and, finally, factor -2 out of the last column, we obtain, using (4.1.10), the representation

$$(4.4.29) \quad \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \Bigg| \begin{bmatrix} y^0 \\ \vdots \\ y^k \end{bmatrix} = -(k!)^{-2}(-\tfrac{1}{2})^k \begin{vmatrix} \|x^i, y^j\|^2 & 1 \\ 1 & 0 \end{vmatrix} ,$$

where $i,j = 0,\ldots,k$, whenever the base space is R_n. This expresses the inner product of two k-cells in terms of a bordered determinant whose entrants are the squares of the ordinary distances of their vertices.

Now we use (4.3.9) to derive the following theorem from (4.4.23) by the same method just used to derive (4.4.29) from (4.4.6).

(4.4.30) <u>Theorem.</u> When the base space is R_n and k is an even integer,

$$\begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \Bigg| \begin{bmatrix} y^0 \\ \vdots \\ y^k \end{bmatrix}^{k-1} = -\frac{(k!)^2}{2^k k^{2k}} \begin{vmatrix} c^i_j & 1 \\ 1 & 0 \end{vmatrix} ,$$

$$\text{where}\quad c^i_j = \left\| \begin{matrix} x^0 \\ \vdots \\ x^{k-i-1} \\ x^{k-i+1} \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^{k-j-1} \\ y^{k-j+1} \\ \vdots \\ y^k \end{matrix} \right\|^2 , \qquad i,j = 0,\ldots,k.$$

Here we have a dual representation of the inner product of two k-cells in R_n to accompany (4.4.29). Here it is expressed in terms of the squares of the "distances" of the faces of the cells whereas in (4.4.29) it is expressed in terms of the squares of the distances of their vertices.

Now, noting that

$$(-1)^{i+j} \left[\begin{matrix} x^0 \\ \vdots \\ x^{k-i-1} \\ x^{k-i+1} \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^{k-j-1} \\ y^{k-j+1} \\ \vdots \\ y^k \end{matrix} \right] = \left[(-1)^i \begin{pmatrix} x^0 \\ \vdots \\ x^{k-i-1} \\ x^{k-i+1} \\ \vdots \\ x^k \end{pmatrix} \middle| (-1)^j \begin{pmatrix} y^0 \\ \vdots \\ y^{k-j-1} \\ y^{k-j+1} \\ \vdots \\ y^k \end{pmatrix} \right],$$

we obtain the following theorem in a manner similar to that by which we arrived at (4.4.30). We use (4.3.9) on (4.4.22).

(4.4.31) <u>Theorem.</u> When the base space is R_n,

$$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix} \right]^{k-1} = \frac{(-)^{k-1}(k!)^2}{2^k k^{2k}(k+1)^2} \begin{vmatrix} c^i_j & 1 \\ 1 & 0 \end{vmatrix} ,$$

$i,j = 0, 1, \ldots, k,$

$$\text{where} \quad c^i_j = \left\| (-)^i \begin{pmatrix} x^0 \\ \vdots \\ x^{k-i-1} \\ x^{k-i+1} \\ \vdots \\ x^k \end{pmatrix} \middle| (-)^j \begin{pmatrix} y^0 \\ \vdots \\ y^{k-j-1} \\ y^{k-j+1} \\ \vdots \\ y^k \end{pmatrix} \right\|^2 .$$

Here we have another dual representation of the inner product of two k-cells to accompany (4.4.29).

Next, we can write (4.4.8) in the equivalent form

$$\begin{bmatrix} x^0 & \Big| & y^0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x^k & \Big| & y^k \end{bmatrix}^{C(k-1,r-1)} = -\frac{(r!)^{2C(k,r)}}{(k!)^{2C(k-1,r-1)}} \begin{vmatrix} 0 & 0 & \cdot & 0 & 1 \\ 0 & d^1_1 & \cdot & d^1_R & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & d^R_1 & \cdot & d^R_R & 1 \\ 1 & 1 & \cdot & 1 & 0 \end{vmatrix}$$

$$\text{where} \quad d^i_j = \begin{bmatrix} x^0 & \Big| & y^0 \\ x^{i_1} & & y^{j_1} \\ \vdots & & \vdots \\ x^{i_r} & \Big| & y^{j_r} \end{bmatrix} \quad \text{and} \quad i,j = 1,\ldots,(R = C(k,r)).$$

From this it follows that

$$\begin{bmatrix} x^0 & \Big| & y^0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x^k & \Big| & y^k \end{bmatrix}^{C(k-1,r-1)} = -\frac{(r!)^{2C(k,r)}}{(k!)^{2C(k-1,r-1)}} \left(-\tfrac{1}{2}\right)^R \begin{vmatrix} 0 & 0 & \cdot & 0 & 1 \\ 0 & -2d^1_1 & \cdot & -2d^1_R & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -2d^R_1 & \cdot & -2d^R_R & 1 \\ 1 & 1 & \cdot & 1 & 0 \end{vmatrix} .$$

If we augment the elements of the i-th row (and the j-th column) by

$$\left\| \begin{matrix} x^0 \\ x^{i_1} \\ \vdots \\ x^{i_r} \end{matrix} \right\|^2 \quad \left(\text{and} \quad \left\| \begin{matrix} y^0 \\ y^{j_1} \\ \vdots \\ y^{j_r} \end{matrix} \right\|^2 \right)$$

times the corresponding elements of the last row (and the last column), then the elements, except those of the first and last rows and columns, of the resulting determinant will have the same form as the right member of (4.3.9). Hence, we have the following theorem.

(4.4.32) <u>Theorem.</u> When the base space is R_n,

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}^{C(k-1,r-1)} = -\frac{(-\frac{1}{2})^{C(k,r)}(r!)^{2C(k,r)}}{(k!)^{2C(k-1,r-1)}} \begin{vmatrix} d^i_j & 1 \\ 1 & 0 \end{vmatrix},$$

where $d^0_0 = 0$, and, for $i,j = 1, \ldots, C(k,r)$,

$$d^i_0 = \left\| \begin{matrix} x^0 \\ x^{i_1} \\ \vdots \\ x^{i_r} \end{matrix} \right\|^2, \quad d^i_j = \left\| \begin{matrix} x^0 & y^0 \\ x^{i_1} & y^{j_1} \\ \vdots & \vdots \\ x^{i_r} & y^{j_r} \end{matrix} \right\|^2, \quad d^0_j = \left\| \begin{matrix} y^0 \\ y^{j_1} \\ \vdots \\ y^{j_r} \end{matrix} \right\|^2 .$$

Here we have a generalization of (4.4.29).

Similarly, we obtain, from (4.4.24), the following theorem, to which (4.4.32) is easily seen to be merely a corollary if we take $\left[x^0 \mid y^0\right] = 1$, $h = 0$, and $0! = 1$.

(4.4.33) <u>Theorem.</u> When the base space is R_n,

$$\begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix}\!\!\begin{vmatrix} y^0 \\ \vdots \\ y^k \end{vmatrix}^P \begin{bmatrix} x^0 \\ \vdots \\ x^h \end{bmatrix}\!\!\begin{vmatrix} y^0 \\ \vdots \\ y^h \end{vmatrix}^Q = -\frac{(-\tfrac{1}{2})^{P+Q}(r!)^{2(P+Q)}}{(k!)^{2P}(h!)^{2Q}} \begin{vmatrix} d^i_j & 1 \\ 1 & 0 \end{vmatrix},$$

where $i,j = 0,1,\ldots,C(k-h,r-h)$, $\quad 0 \leq h < r$,

$P = C(k-h-1,r-h-1)$, $\quad Q = C(k-h-1,r-h)$,

and

$$d^i_0 = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{i_1} \\ \vdots \\ x^{i_{r-h}} \end{matrix} \right\|^2, \quad d^i_j = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{i_1} \\ \vdots \\ x^{i_{r-h}} \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^h \\ y^{j_1} \\ \vdots \\ y^{j_{r-h}} \end{matrix} \right\|^2, \quad d^0_j = \left\| \begin{matrix} y^0 \\ \vdots \\ y^h \\ y^{j_1} \\ \vdots \\ y^{j_{r-h}} \end{matrix} \right\|^2.$$

Here we have a further generalization of (4.4.29).

Let us recall (4.4.25). In $D_r(d)$, let us write

$(-)^{i_{k-r}+j_{k-r}}\, d^{i_{k-r}}_{j_{k-r}}$ in the form

$$\left[(-)^{i_{k-r}} \begin{pmatrix} x^{(i_1)\cdots(i_{k-r})_0} \\ \vdots \\ x^{(i_1)\cdots(i_{k-r})_r} \end{pmatrix} \middle| (-)^{j_{k-r}} \begin{pmatrix} y^{(j_1)\cdots(j_{k-r})_0} \\ \vdots \\ y^{(j_1)\cdots(j_{k-r})_r} \end{pmatrix} \right],$$

which we can do by (4.2.3) and (4.2.11). Then, if we take the base space to be R_n, we may apply (4.3.9); and, as in (4.4.31), derive

$$D_r(d) = (-\tfrac{1}{2})^{\frac{r+1}{r}} \begin{vmatrix} c_{j_{k-r}}^{i_{k-r}} & 1 \\ 1 & 0 \end{vmatrix}^{1/r} ,$$

where

$$c_{j_{k-r}}^{i_{k-r}} = \left\| (-)^{i_{k-r}} \begin{pmatrix} x^{(i_1)\cdots(i_{k-r})_0} \\ \vdots \\ x^{(i_1)\cdots(i_{k-r})_r} \end{pmatrix} \middle| (-)^{j_{k-r}} \begin{pmatrix} y^{(j_1)\cdots(j_{k-r})_0} \\ \vdots \\ y^{(j_1)\cdots(j_{k-r})_r} \end{pmatrix} \right\|^2 .$$

If we denote this new form of $D_r(d)$ by $(-\tfrac{1}{2})^{(r+1)/r} \underline{D}_r(d)$, we can write (4.4.25) in the form

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix} = B_r D_{k-1}(\cdot\ \cdot\ \cdot\ D_{r+1}((-\tfrac{1}{2})^{\frac{r+1}{r}} \underline{D}_r(c)).\ .\ .).$$

Advancing the coefficient of $\underline{D}_r(d)$ to the position of coefficient of D_{k-1} as we advanced A_r in the proof of (4.4.25), we obtain

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix} = (-\tfrac{1}{2})^{\frac{k}{r}} B_r D_{k-1}(\cdot\ \cdot\ \cdot\ D_{r+1}(\underline{D}_r(c)).\ .\ .).$$

If we denote the right member of this equation by

$C_{rk}\left\{\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}\right\}$, we have the following theorem.

(4.4.34) <u>Theorem.</u> When the base space is R_n,

$$\begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \begin{vmatrix} y^0 \\ \vdots \\ y^k \end{vmatrix} = c_{rk} \left\{ \begin{bmatrix} x^0 \\ \vdots \\ x^k \end{bmatrix} \begin{vmatrix} y^0 \\ \vdots \\ y^k \end{vmatrix} \right\}, \qquad 0 < r < k.$$

This expresses the inner product of two k-cells in terms of the squares of the "distances" of the r-dimensional faces of the faces . . . of the faces of one and those of the other.

Now we have

$$(1/k!)^2 \left\| \begin{matrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{matrix} \right\| \left\| \begin{matrix} 1 & y^0_{j_1} & \cdot & y^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & y^k_{j_1} & \cdot & y^k_{j_k} \end{matrix} \right\|^*$$

$$= -(1/k!)^2 \left\| \begin{matrix} x^0_{j_1} & \cdot & x^0_{j_k} & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^k_{j_1} & \cdot & x^k_{j_k} & 1 & 0 \\ 0 & \cdot & 0 & 0 & 1 \end{matrix} \right\| \left\| \begin{matrix} y^0_{j_1} & \cdot & y^0_{j_k} & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y^k_{j_1} & \cdot & y^k_{j_k} & 0 & 1 \\ 0 & \cdot & 0 & 1 & 0 \end{matrix} \right\|^*$$

$$= -(1/k!)^2 \left| \begin{matrix} (x^0,y^0)_j & \cdot & (x^0,y^k)_j & 1 \\ \cdot & \cdot & \cdot & \cdot \\ (x^k,y^0)_j & \cdot & (x^k,y^k)_j & 1 \\ 1 & 1 & 1 & 0 \end{matrix} \right|$$

$$= -(1/k!)^2 \begin{vmatrix} (x^u, y^v)_j & 1 \\ 1 & 0 \end{vmatrix}, \quad (x^u, y^v)_j = \sum_{t=1}^{k} x^u_{j_t} \overline{y^v_{j_t}} \quad .$$

Hence, by (4.1.15), we have the following representation

$$(4.4.35) \quad \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = -(1/k!)^2 \sum_{j=1}^{C(n,k)} \begin{vmatrix} (x^u, y^v)_j & 1 \\ 1 & 0 \end{vmatrix} ,$$

where $u,v = 0,1,\dots,k$ and $(x^u,y^v)_j = \sum_{t=1}^{k} x^u_{j_t} \overline{y^v_{j_t}}$.

Now, in view of (4.4.6), we have

$$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = \sum_{j=1}^{C(n,k)} \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right]_j ,$$

where the subscript j indicates that the inner product is to be taken over the projections of $((x^0,\dots,x^k))$ and $((y^0,\dots,y^k))$ on the j-th k-dimensional coordinate space, the ordering being lexicographical. If moreover we write (4.4.25) in the form

$$\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] = S_{rk} \left\{ \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} y^0 \\ \vdots \\ y^k \end{matrix}\right] \right\} ,$$

then, as (4.4.25) was derived from (4.4.6), so can the following representation be derived from (4.4.35).

$$(4.4.36)\quad \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix} = \sum_{j=1}^{C(n,k)} S_{rk}\left\{ \begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix}_j \right\}, \qquad 0 < r < k.$$

By the same method used to derive (4.4.34) from (4.4.25), we derive the next theorem from (4.4.36).

(4.4.37) <u>Theorem.</u> When the base space is R_n,

$$\begin{bmatrix} x^0 & y^0 \\ \vdots & \vdots \\ x^k & y^k \end{bmatrix} = \sum_{j=1}^{C(n,k)} C_{rk}\left\{ \begin{bmatrix} x^0 & y^0 \\ x^k & y^k \end{bmatrix}_j \right\}, \qquad 0 < r < k.$$

The last theorem expresses the inner product of $((x^0,\ldots,x^k))$ with $((y^0,\ldots,y^k))$ in terms of the "distances" of the r-dimensional faces of the faces . . . of the faces of the components of $((x^0,\ldots,x^k))$ and those of $((y^0,\ldots,y^k))$. Whereas, the preceding theorem expresses the inner product of $((x^0,\ldots,x^k))$ with $((y^0,\ldots,y^k))$ as a function of the inner products of the r-dimensional faces of the faces . . . of the faces of the components of $((x^0,\ldots,x^k))$ with those of $((y^0,\ldots,y^k))$.

<u>4.5. Euclidean Volume of k-Cells.</u> When the base space is R_n, it can be shown that the norm of a k-cell is just its k-dimensional Euclidean volume. If we denote the "volume" of the k-cell $((x^0,\ldots,x^k))$ by V_k, then we can express V_k in the following divers forms.

From (4.1.15) and (4.1.16), we have, in R_n,

$$(4.5.1) \qquad V_k^2 = (1/k!)^2 \sum_{j=1}^{C(n,k)} \begin{vmatrix} 1 & x_{j_1}^0 & \cdot & x_{j_k}^0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x_{j_1}^k & \cdot & x_{j_k}^k \end{vmatrix}^2 ,$$

which expresses the "volume" in terms of the squares of the "volumes" of the k-cells which are the perpendicular projections of $((x^0,\ldots,x^k))$ on the k-dimensional coordinate linear manifolds. This expression for the "volume" is noted in [11, p. 80].

From (4.4.3), we have

$$(4.5.2) \qquad V_k^2 = (1/k!)^2 \left| XX' \right| ,$$

where X is the matrix of edges at the initial vertex of $((x^0,\ldots,x^k))$. This expression for the "volume" is established in [2, p. 296, Ex. 2(c)].

From (4.4.4), we have

$$(4.5.3) \qquad V_k^2 = (1/k!)^2 \left| (x^i - x^0,\ x^j - x^0) \right| , \qquad i,j = 1,\ldots,k,$$

which expresses the volume in terms of the inner products of the edges issuing from the initial vertex. The determinant in (4.5.3) is known as Gram's determinant and plays an important role in Mathematics, see [4] and [13].

From (4.4.6), we have

$$(4.5.4) \quad V_k^2 = -(1/k!)^2 \begin{vmatrix} (x^i, x^j) & 1 \\ 1 & 0 \end{vmatrix}, \qquad i,j = 0,\dots,k,$$

which expresses the "volume" in terms of the inner products of the vectors $x^0, x^1, \dots, x^k$ which determine the vertices of $((x^0,\dots,x^k))$.

From (4.4.24), we have

$$(4.5.5) \qquad V_k^{2P} = \frac{(r!)^{2(P+Q)}}{(k!)^{2P}(h!)^{2Q}} \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \end{matrix} \right\|^{-2Q} \left| b_j^i \right| ,$$

where

$$b_j^i = \left[\begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{i_1} \\ \vdots \\ x^{i_{r-h}} \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{j_1} \\ \vdots \\ x^{j_{r-h}} \end{matrix} \right], \qquad i,j = 1,\dots,C(k-h,r-h),$$

$P = C(k-h-1,\ r-h-1)$, $\quad Q = C(k-h-1,\ r-h)$, and the order is lexicographical order.

This expresses the "volume" in terms of the generalized inner products of the r-cells of $((x^0,\dots,x^k))$ which have a common initial h-cell with $((x^0,\dots,x^k))$ and the volume of the common h-cell. In particular, for $h = 0$, we take $0! = 1$, $\|x^0\|^{-2Q} = 1$, and so obtain

$$(4.5.5.1) \qquad V_k^{2C(k-1,\ r-1)} = \frac{(r!)^{2C(k,r)}}{(k!)^{2C(k-1,\ r-1)}} \left| b_j^i \right| ,$$

where $b^i_j = \begin{bmatrix} x^0 & x^0 \\ x^{i_1} & x^{j_1} \\ \vdots & \vdots \\ x^{i_r} & x^{j_r} \end{bmatrix}$ in lexicographical order.

From (4.4.22), we have

$$(4.5.6) \qquad V_k^{2(k-1)} = -\frac{(k!)^2}{k^{2k}(k+1)^2} \begin{vmatrix} (-)^{i+j} c^i_j & 1 \\ 1 & 0 \end{vmatrix},$$

where $c^i_j = \begin{bmatrix} x^{i_1} & x^{j_1} \\ \vdots & \vdots \\ x^{i_k} & x^{j_k} \end{bmatrix}$, and the order is lexicographical order. This expresses the "volume" in terms of the inner products of the (k-1)-cells determined by the vertices of $((x^0,\ldots,x^k))$, i.e., in terms of the inner products of its faces.

From (4.4.29), we have

$$(4.5.7) \qquad V_k^2 = -(-\tfrac{1}{2})^k (1/k!)^2 \begin{vmatrix} \|x^i, x^j\|^2 & 1 \\ 1 & 0 \end{vmatrix},$$

where $i,j = 0,1,\ldots,k$. This expresses the "volume" in terms of the squares of the distances of the vertices. In view of (4.1.11) and (4.1.12), the determinant in (4.5.7) is seen to be symmetric with zero's along the principal diagonal.

From (4.4.33), we have

$$(4.5.8)\quad V_k^{2P} V_h^{2Q} = -\frac{(-\frac{1}{2})^{P+Q} (r!)^{2(P+Q)}}{(k!)^{2P} (h!)^{2Q}} \begin{vmatrix} 0 & d_1^0 & \cdot & d_R^0 & 1 \\ d_0^1 & 0 & \cdot & d_R^1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_0^R & d_1^R & \cdot & 0 & 1 \\ 1 & 1 & \cdot & 1 & 0 \end{vmatrix}$$

where $P = C(k-h-1,\ r-h-1)$, $Q = C(k-h-1,\ r-h)$, $R = C(k-h, r-h)$,

$$d_0^i = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{i_1} \\ \vdots \\ x^{i_{r-h}} \end{matrix} \right\|^2, \quad d_j^i = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{i_1} \\ \vdots \\ x^{i_{r-h}} \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{j_1} \\ \vdots \\ x^{j_{r-h}} \end{matrix} \right\|^2, \quad d_j^0 = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{j_1} \\ \vdots \\ x^{j_{r-h}} \end{matrix} \right\|^2,$$

$V_h = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \end{matrix} \right\|$, and $0 \leqq h < r$. This expresses the "volume" in terms of the "distances" of the r-cells issuing from the initial h-cell and the "volumes" of those r-cells and the h-cell. When $h = 0$ and $r = 1$, (4.5.8) reduces to (4.5.7), and, hence, is a generalization thereof.

From (4.4.25), we can obtain a representation, of which (4.5.6) is a particular case, which expresses the square of the "volume" of a k-cell in terms of the inner products of the r-dimensional faces of the faces of the faces . . . of the faces of the k-cell.

From (4.4.27) and (4.4.28), we obtain respectively

$$(4.5.9) \qquad V_k^2 = \sum_{j=1}^{C(n,k)} \left| E_r(X_{kn}^j) \right|^2$$

and

$$(4.5.10) \qquad V_k^2 = \sum_{j=1}^{C(n,k)} \left| F_r(X_{kn}^j) \right|^2 \quad ,$$

where the vertical bars denote the absolute value. (4.5.9) expresses the "volume" of a k-cell in terms of the components of the r-dimensional faces of the faces . . . of the faces of the k-cell. Whereas (4.5.10) expresses it in terms of the components of the r-dimensional faces of the faces . . . of the faces of the k-cell, all having a common initial vertex.

From (4.4.26), we have the following interesting relation.

$$(4.5.11) \qquad \left| \left[(x^i - x^0,\ x^j - x^0) \right] + I_k \right| = \sum_{r=0}^{k} \sum_{\nu=1}^{C(k,r)} (r!)^2 \left\| \begin{matrix} x^{v_0} \\ \vdots \\ x^{v_r} \end{matrix} \right\|^2$$

where $v_0 = 0$, $i,j = 1,\ldots,k$, and the order is lexicographical order. Here we have an expression involving the volumes of all r-cells, $r = 0,1,\ldots,k$, determined by the vertices of, and having a common initial vertex with, $((x^0,\ldots,x^k))$.

Using the method employed to derive (4.4.29) from (4.4.4), by way of (4.4.6), we derive, from (4.5.11), when the base space is R_n,

$$(4.5.12) \qquad -(-2)^k \sum_{r=0}^{k} \sum_{\nu=1}^{C(k,r)} (r!) \left\| \begin{matrix} x^{v_0} \\ \vdots \\ x^{v_r} \end{matrix} \right\|^2$$

$$= \begin{vmatrix} 0 & \|x^0,x^1\|^2 & . & \|x^0,x^k\|^2 & 1 \\ \|x^1,x^0\|^2 & -2 & . & \|x^1,x^k\|^2 & 1 \\ . & . & . & . & . \\ \|x^k,x^0\|^2 & \|x^k,x^1\|^2 & . & -2 & 1 \\ 1 & 1 & . & 1 & 0 \end{vmatrix}$$

where $v_0 = 0$ and the ordering is lexicographical. We have here a relation involving the "volumes" of the r-cells of $((x^0,\ldots,x^k))$, having a common initial vertex, and the distances of the vertices of $((x^0,\ldots,x^k))$.

In a similar manner, it can be shown that, when the base is R_n,

$$(4.5.13) \qquad \begin{vmatrix} b_j^i & 1 \\ 1 & 0 \end{vmatrix} = -(-2)^{k-h} \sum_{r=0}^{k-h} S_r,$$

where $i,j = 0,1,\ldots,k-h,$ $\quad b_0^0 = 0,$ $\quad b_i^i = -2,$

$$b_0^i = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{h+i} \end{matrix} \right\|^2, \qquad b_j^0 = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{h+j} \end{matrix} \right\|^2 \qquad ,$$

$$b^i_j = \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{h+i} \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{h+j} \end{matrix} \right\| \qquad \text{when } i \neq j, \quad S_0 = 1,$$

and, for $0 < r \leqq k$,

$$S_r = \sum_{\nu=1}^{C(k-h,r)} \frac{((h+r)!)^2}{(h!)^2 (h+1)^{2r}} \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \end{matrix} \right\|^{2r-2} \left\| \begin{matrix} x^0 \\ \vdots \\ x^h \\ x^{h+v_1} \\ \vdots \\ x^{h+v_r} \end{matrix} \right\|^2 ,$$

$(v_1, \ldots, v_r)$ runs over the r-tuples of $1, 2, \ldots, k-h$ lexicographically.

The left member of (4.5.13) involves the distances of the (h+1)-cells having a common initial h-cell, except for the first and last rows and columns; whereas the right member involves the squares of the volumes of all cells of $((x^0, \ldots, x^k))$ having a common initial h-cell. Hence (4.5.13) is a natural extension of (4.5.12).

When the base space is R_n, the signed "volume" V_n of $((x^0, \ldots, x^n))$ is known to be

$$(4.5.14) \qquad V_n = (1/n!) \left| 1 \quad x^i_j \right|, \qquad \begin{matrix} i = 0, \ldots, n, \\ j = 1, \ldots, n. \end{matrix}$$

More generally, taking $k = n$ in (3.5.1), we have

$$(4.5.14) \qquad V_n = E_r(X_{nn}),$$

which expresses the "volume" of $((x^0, \ldots, x^n))$ in terms of the components of the r-dimensional faces of the faces . . . of the faces of $((x^0, \ldots, x^n))$, if we admit a possible occurrence of certain roots of unity.

Similary, if we take $k = n$ in (3.5.2), we have

$$(4.5.15) \qquad V_n = F_r(X_{nn}),$$

which expresses the "volume" of $((x^0, \ldots, x^n))$ in terms of the components of its r-cells having a common initial vertex, if we admit the possible occurrence of certain roots of unity.

Finally, it follows from (4.4.34), (4.4.35), (4.4.36), (4.4.37) respectively that the next four relations are true.

$$(4.5.16) \qquad V_k^2 = C_{rk}\left\{\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\right]\right\}, \qquad 0 < r < k \quad (\text{in } R_n).$$

$$(4.5.17) \qquad V_k^2 = -(1/k!)^2 \sum_{j=1}^{C(n,k)} \begin{vmatrix} (x^u, x^v)_j & 1 \\ 1 & 0 \end{vmatrix} .$$

$$(4.5.18) \qquad V_k^2 = \sum_{j=1}^{C(n,k)} S_{rk}\left\{\left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix}\right]_j\right\}, \qquad 0 < r < k.$$

$$(4.5.19) \qquad V_k^2 = \sum_{j=1}^{C(n,k)} C_{rk} \left\{ \left[\begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \middle| \begin{matrix} x^0 \\ \vdots \\ x^k \end{matrix} \right] \right\}, \quad 0 < r < k \quad (\text{in } R_n).$$

These four relations may be interpreted as follows. (4.5.16) expresses the "volume" as a function of the "distances" of the r-dimensional faces of the faces . . . of the faces of the cell; (4.5.17) expresses it as a function of the ordinary inner products of the vectors which determine the projections of the k-cell on the k-dimensional coordinate spaces; (4.5.18) expresses it as a function of the generalized inner products of the r-dimensional faces of the faces . . . of the faces of the component cells; finally, (4.5.19) expresses it in terms of the "distances" of the faces of the faces . . . of the faces of the components.

Let us take $k = n = 3$, $(x^i-x^0, x^i-x^0) = \|x^0,x^i\|^2$, and $(x^i-x^0, x^j-x^0) = \|x^0,x^i\|\cdot\|x^0,x^j\| \cos \measuredangle\, x^ix^0x^j$ in (4.5.3). Then we see by expanding the right member of (4.5.3) that the following remark is true.

(4.5.20) <u>Remark.</u> Euler's formula [6] for the volume of a tetrahedron follows from (4.5.3) as a corollary.

Chapter 5

Properties of k-Cell Components: Associated N-Space

It follows from (3.3.3) that any k-cell $((x^0, \ldots, x^k))$ in $L_n(K)$ has a set of components

$$(\ldots, (1/k!)\underline{X}_{j_1 \ldots j_k}, \ldots),$$

where

$$\underline{X}_{j_1 \ldots j_k} = \begin{vmatrix} 1 & x^0_{j_1} & \cdot & x^0_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix},$$

which are themselves elements of K; but it does not follow that any set of $C(n,k)$ elements of K are necessarily the components of a k-cell. In fact, this is not true unless $k = 1$ or $k = n-1$. It is therefore pertinent to determine what, if any, algebraic relations connect the components of a k-cell. It is obvious that the components of every k-cell are skew-symmetric in their suffixes; for it follows from the elementary properties of determinants that

$$\underline{X}_{j_1 \ldots j_k} = 0$$

if the integers $j_1, \ldots, j_k$ are not all distinct and that, if $i_1, \ldots, i_k$ is any permutation of the set $j_1, \ldots, j_k$,

$$\underline{X}_{i_1 \cdots i_k} = \epsilon \, \underline{X}_{j_1 \cdots j_k} ,$$

where $\epsilon = +1$ or -1 according as the permutation is even or odd.

5.1. The Grassmann Quadratic Relations. The arguments in this section are essentially those made in [8, pp. 309-315]. We first note that there can be no linear relation. For let

$$\sum_{i=1}^{C(n,k)} a_{i_1 \cdots i_k} Z_{i_1 \cdots i_k}$$

be a homogeneous linear form over K in the $C(n,k)$ indeterminates $Z_{i_1 \cdots i_k}$ (which are skew-symmetric in their suffixes) which vanishes whenever the $Z_{i_1 \cdots i_k}$ are replaced by the corresponding components $\underline{X}_{i_1 \cdots i_k}$ of any k-cell. Let $(1/k!)\, \underline{X}^{j_1 \cdots j_k}_{i_1 \cdots i_k}$ denote the i-th component of the k-cell $((\theta, e^{j_1}, \ldots, e^{j_k}))$ determined by the null vector and the j-th k-tuple of the n unit basis vectors for $L_n(K)$ taken in lexicographical order. Clearly

$$\underline{X}^{j_1 \cdots j_k}_{i_1 \cdots i_k} = 0$$

unless $i_1, \ldots, i_k$ is a permutation of $j_1, \ldots, j_k$, while

$$\underline{X}^{j_1 \cdots j_k}_{j_1 \cdots j_k} = 1 .$$

We have therefore, after factoring out $1/k!$,

$$0 = \sum_{i=1}^{C(n,k)} a_{i_1 \cdots i_k} \underline{X}_{i_1 \cdots i_k}^{j_1 \cdots j_k} = a_{j_1 \cdots j_k} \quad .$$

Since this is true for every k-tuple $j_1 \cdots j_k$ of the integers $1, \ldots, n$, we have

(5.1.1) <u>Theorem.</u> The components of the proper k-cells in $L_n(K)$ do not satisfy any linear relation of the form

$$\sum_{i=1}^{C(n,k)} a_{i_1 \cdots i_k} \underline{X}_{i_1 \cdots i_k} = 0.$$

On the other hand, the homogeneous quadratic form

$$Z_{i_1 \cdots i_k} Z_{j_1 \cdots j_k} - \sum_{s=1}^{k} Z_{i_1 \cdots i_{r-1} j_s i_{r+1} \cdots i_k} Z_{j_1 \cdots j_{s-1} i_r j_{s+1} \cdots j_k}$$

over K in the $C(n,k)$ indeterminates $Z_{i_1 \cdots i_k}$ (which are skew-symmetric in their suffixes) vanishes whenever the indeterminates $Z_{i_1 \cdots i_k}$ are replaced by the corresponding components $(1/k!)\underline{X}_{i_1 \cdots i_k}$ of any k-cell in $L_n(K)$. For, in view of the property of skew-symmetry, we have then

$$1/k!(\underline{X}_{i_1 \cdots i_k} \underline{X}_{j_1 \cdots j_k}$$

$$- \sum_{s=1}^{k} (-1)^{k-s} \underline{X}_{i_1 \cdots i_{r-1} j_s i_{r+1} \cdots i_k} \underline{X}_{j_1 \cdots j_{s-1} j_{s+1} \cdots j_k i_r}).$$

Now let $X_{i_1 \cdots i_k}$ denote the determinant of the $k \times k$ matrix whose u-th column ($u = 1, \ldots, k$) is just the i_u-th column of the matrix of edges at the initial vertex of $((x^0, \ldots, x^k))$. Then, since

$$\begin{vmatrix} 1 & x^0_{i_1} & \cdot & x^0_{i_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{i_1} & \cdot & x^k_{i_k} \end{vmatrix} = \begin{vmatrix} x^1_{i_1} - x^0_{i_1} & \cdot & x^1_{i_k} - x^0_{i_k} \\ \cdot & \cdot & \cdot \\ x^k_{i_1} - x^0_{i_1} & \cdot & x^k_{i_k} - x^0_{i_k} \end{vmatrix} ,$$

it is clear that

$$\underline{X}_{i_1 \cdots i_k} = X_{i_1 \cdots i_k} .$$

Let, further, $\widetilde{x^u_{i_r} - x^0_{i_r}}$ denote the cofactor of $x^u_{i_r} - x^0_{i_r}$ in $X_{i_1 \cdots i_k}$. Now we can write

$$\sum_{s=1}^{k} (-1)^{k-s} \underline{X}_{i_1 \cdots i_{r-1} j_s i_{r+1} \cdots i_k} \underline{X}_{j_1 \cdots j_{s-1} j_{s+1} \cdots j_k i_r}$$

$$= \sum_{s=1}^{k} (-1)^{k-s} X_{i_1 \cdots i_{r-1} j_s i_{r+1} \cdots i_k} X_{j_1 \cdots j_{s-1} j_{s+1} \cdots j_k i_r}$$

$$= \sum_{s=1}^{k} (-1)^{k-s} \left(\sum_{u=1}^{k} (x^u_{j_s} - x^0_{j_s}) \left(\widetilde{x^u_{i_r} - x^0_{i_r}}\right)\right) X_{j_1 \cdots j_{s-1} j_{s+1} \cdots j_k i_r}$$

$$= -\sum_{u=1}^{k} \left(\widetilde{x^u_{i_r} - x^0_{i_r}}\right) \sum_{s=1}^{k} \begin{vmatrix} x^1_{j_1} - x^0_{j_1} & \cdot & x^1_{j_s} - x^0_{j_s} & \cdot & x^1_{j_k} - x^0_{j_k} & x^1_{i_r} - x^0_{i_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x^k_{j_1} - x^0_{j_1} & \cdot & x^k_{j_s} - x^0_{j_s} & \cdot & x^k_{j_k} - x^0_{j_k} & x^k_{i_r} - x^0_{i_r} \\ 0 & \cdot & x^u_{j_s} - x^0_{j_s} & \cdot & 0 & 0 \end{vmatrix}$$

$$= -\sum_{u=1}^{k} \widetilde{(x^u_{i_r} - x^0_{i_r})} \begin{vmatrix} x^1_{j_1} - x^0_{j_1} & \cdot & x^1_{j_k} - x^0_{j_k} & x^1_{i_r} - x^0_{i_r} \\ \cdot & \cdot & \cdot & \cdot \\ x^k_{j_1} - x^0_{j_1} & \cdot & x^k_{j_k} - x^0_{j_k} & x^k_{i_r} - x^0_{i_r} \\ x^u_{j_1} - x^0_{j_1} & \cdot & x^u_{j_k} - x^0_{j_k} & 0 \end{vmatrix}$$

$$= \sum_{u=1}^{k} \widetilde{(x^u_{i_r} - x^0_{i_r})} \begin{vmatrix} x^1_{j_1} - x^0_{j_1} & \cdot & x^1_{j_k} - x^0_{j_k} & x^1_{i_r} - x^0_{i_r} \\ \cdot & \cdot & \cdot & \cdot \\ x^k_{j_1} - x^0_{j_1} & \cdot & x^k_{j_k} - x^0_{j_k} & x^k_{i_r} - x^0_{i_r} \\ 0 & \cdot & 0 & x^u_{i_r} - x^0_{i_r} \end{vmatrix}$$

$$= \sum_{u=1}^{k} \widetilde{(x^u_{i_r} - x^0_{i_r})} \, (x^u_{i_r} - x^0_{i_r}) \, X_{j_1 \ldots j_k}$$

$$= X_{i_1 \ldots i_k} X_{j_1 \ldots j_k}$$

$$= \underline{X}_{i_1 \ldots i_k} \underline{X}_{j_1 \ldots j_k} \quad ,$$

where we first replaced the $\underline{X}_{i_1 \ldots i_k}$ by the equivalent $X_{i_1 \ldots i_k}$, then expanded $X_{i_1 \ldots i_{r-1} j_s i_{r+1} \ldots i_k}$ in terms of the elements and their cofactors of the r-th column, reversed the order of summation, wrote $(x^u_{j_s} - x^0_{j_s})$ times $X_{j_1 \ldots j_{s-1} j_{s+1} \ldots j_k i_r}$ in the equivalent form

$$(-1)^{k+s+1}\begin{vmatrix} x^1_{j_1}-x^0_{j_1} & \cdot & x^1_{j_s}-x^0_{j_s} & \cdot & x^1_{j_k}-x^0_{j_k} & x^1_{i_r}-x^0_{i_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x^k_{j_1}-x^0_{j_1} & \cdot & x^k_{j_s}-x^0_{j_s} & \cdot & x^k_{j_k}-x^0_{j_k} & x^k_{i_r}-x^0_{i_r} \\ 0 & \cdot & x^u_{j_s}-x^0_{j_s} & \cdot & 0 & 0 \end{vmatrix}$$

summed (over s) the k determinants (which differed only in the last row), replaced

$$\begin{vmatrix} x^1_{j_1}-x^0_{j_1} & \cdot & x^1_{j_k}-x^0_{j_k} & x^1_{i_r}-x^0_{i_r} \\ \cdot & \cdot & \cdot & \cdot \\ x^k_{j_1}-x^0_{j_1} & \cdot & x^k_{j_k}-x^0_{j_k} & x^k_{i_r}-x^0_{i_r} \\ x^u_{j_1}-x^0_{j_1} & \cdot & x^u_{j_k}-x^0_{j_k} & 0 \end{vmatrix}$$

by

$$-\begin{vmatrix} x^1_{j_1}-x^0_{j_1} & \cdot & x^1_{j_k}-x^0_{j_k} & x^1_{i_r}-x^0_{i_r} \\ \cdot & \cdot & \cdot & \cdot \\ x^k_{j_1}-x^0_{j_1} & \cdot & x^k_{j_k}-x^0_{j_k} & x^k_{i_r}-x^0_{i_r} \\ 0 & \cdot & 0 & x^u_{i_r}-x^0_{i_r} \end{vmatrix},$$

since

$$\begin{vmatrix} x^1_{j_1}-x^0_{j_1} & \cdot & x^1_{j_k}-x^0_{j_k} & x^1_{i_r}-x^0_{i_r} \\ \cdot & \cdot & \cdot & \cdot \\ x^k_{j_1}-x^0_{j_1} & \cdot & x^k_{j_k}-x^0_{j_k} & x^k_{i_r}-x^0_{i_r} \\ x^u_{j_1}-x^0_{j_1} & \cdot & x^u_{j_k}-x^0_{j_k} & x^u_{i_r}-x^0_{i_r} \end{vmatrix}$$

clearly vanishes by virtue of having two rows alike for every u in the summation, and followed an obvious procedure from there on. We have therefore proved that

$$\underline{X}_{i_1\cdots i_k}\underline{X}_{j_1\cdots j_k}$$

$$- \sum_{s=1}^{k} \underline{X}_{i_1\cdots i_{r-1}j_s i_{r+1}\cdots i_k}\underline{X}_{j_1\cdots j_{s-1}i_r j_{s+1}\cdots j_k}$$

$$= \underline{X}_{i_1\cdots i_k}\underline{X}_{j_1\cdots j_k} - \underline{X}_{i_1\cdots i_k}\underline{X}_{j_1\cdots j_k}$$

$$= 0 .$$

This completes the proof. Consequently, we have the following theorem.

(5.1.2) Theorem. The components of every k-cell $((x^0,\ldots,x^k))$ in $L_n(K)$ satisfy the Grassmann quadratic relations; namely, if $0 < r \leqq k$, then

$$\underline{X}_{i_1\cdots i_k}\underline{X}_{j_1\cdots j_k}$$

$$= \sum_{s=1}^{k} \underline{X}_{i_1\cdots i_{r-1}j_s i_{r+1}\cdots i_k}\underline{X}_{j_1\cdots j_{s-1}i_r j_{s+1}\cdots j_k} .$$

We should remark that (5.1.2) is actually just a corollary to Sylvester's multiplication theorem for determinants, see [11, p. 80].

Next we make the following remark.

(5.1.3) Remark. The relation (5.1.2) is illusory unless the sets $\{i_1, \ldots, i_k\}, \{j_1, \ldots, j_k\}$ differ by at least four integers.

The next remark is an immediate consequence of (5.1.3).

(5.1.4) Remark. The Grassmann quadratic relations are automatically satisfied by an arbitrary set of $C(n,k)$ elements $\underline{X}_{i_1 \ldots i_k}$ (which are skew-symmetric in their suffixes) of K whenever $k = 1$ or $k = n-1$, and trivially so when $k = 0$ or $k = n$.

5.2. k-Cells With Prescribed Components. Let us now settle the question of whether we can prescribe the components of a k-cell before knowing its vertices. We treat the special cases first.

In case $k = 0$, there is $C(n,0) = 1$ component which is, by definition, unity for every 0-cell; hence, if we prescribe unity as the component, any 0-cell (point) in $L_n(K)$ will suffice--in particular, $((\theta))$ is a solution.

In case $k = 1$, there are $C(n,1) = n$ components which may be chosen in any manner from K; for example, let them be $x_1, x_2, \ldots, x_n$. Then the 1-cell $((\theta, x^1))$,

where

$$x^1 = \sum_{i=1}^{k} x_i e^i , \qquad \text{see } (3.1.2),$$

has the prescribed components.

In case $k = n > 0$, there is $C(n,n) = 1$ component. If we prescribe x_0 --any element of K--as the component, then the n-cell $((\theta, x_0 e^1, 2e^2, \ldots, ne^n))$ has the prescribed component.

However, in general, the problem is not quite so simple. In fact, for the remaining cases, it is necessary that K be algebraically closed to assure a solution when k is an even natural number and $1 < k < n$. Even then there exist solutions under certain conditions; namely when the prescribed components are $C(n,k)$ in number, skew-symmetric in their suffixes, and satisfy the Grassmann quadratic relations (5.1.2), which we have already shown to be necessary conditions. It turns out that these conditions are not only necessary but also sufficient conditions that $C(n,k)$ elements--let them be

$$(\ldots, p_{j_1 \ldots j_k}, \ldots)$$

--of K be the components of a k-cell, $0 < k \leq n$, in $L_n(K)$. Of course, as we remarked in (5.1.4), the Grassmann quadratic relations impose no restriction on our choice of components when $k = 0, 1, n-1,$ or n. Now we are ready to state the answer to the question of when there exist k-cells with prescribed components.

(5.2.1) Theorem. If K is a commutative field, without characteristic, which is algebraically closed, $0 < k \leqq n$, and $(\ldots, p_{j_1\ldots j_k}, \ldots)$ are $C(n,k)$ elements of K, which are skew-symmetric in their suffixes and which satisfy the Grassmann quadratic relations, then there is at least one k-cell, with initial vertex at the origin, in $L_n(K)$ which has components $(\ldots, p_{j_1\ldots j_k}, \ldots)$.

We have already seen that (5.2.1) is true for $k = 1, n$. It is obviously true if the prescribed components are all zero, for then any null k-cell with vertex at the origin will suffice. Hence there is no loss of generality in assuming that some prescribed component, say $p_{i_1\ldots i_k}$, is different from zero. Before proving (5.2.1), we prove the following helpful lemma.

(5.2.2) Lemma. The relation

$$p^k_{i_1\ldots i_k} p_{j_1\ldots j_k} = p_{i_1\ldots i_k} \left| b^r_s \right|, \quad r, s = 1, \ldots, k,$$

where $b^r_s = p_{i_1\ldots i_{r-1} j_s i_{r+1}\ldots i_k}$ is the r, s-th element of the determinant on the right, holds for every pair $p_{i_1\ldots i_k}$, $p_{j_1\ldots j_k}$ of any $C(n,k)$-tuple of elements of K, which are skew-symmetric in their suffixes and which satisfy the Grassmann quadratic relations.

The lemma is trivially true for $k = 1$, for then we have simply

$$p_{i_1} p_{j_1} = p_{i_1} \left\| b_1^1 \right\| = p_{i_1} \ p_{j_1} \quad .$$

We remark further that the determinant in the right member of (5.2.2) takes a particularly simple form when $p_{j_1 \cdots j_k}$ is replaced by $p_{i_1 \cdots i_k}$; namely, it becomes a diagonal determinant. We have then

$$p^k_{i_1 \cdots i_k} p_{i_1 \cdots i_k} = p_{i_1 \cdots i_k} \begin{vmatrix} p_{i_1 \cdots i_k} & & 0 \\ & \ddots & \\ 0 & & p_{i_1 \cdots i_k} \end{vmatrix}$$

$$= p_{i_1 \cdots i_k} p^k_{i_1 \cdots i_k} \quad .$$

To prove (5.2.2), we make use of the Grassmann quadratic relations k times in succession as follows.

$$p^k_{i_1 \cdots i_k} p_{j_1 \cdots j_k} = p^{k-1}_{i_1 \cdots i_k} \ p_{j_1 \cdots j_k} \ p_{i_1 \cdots i_k}$$

$$= p^{k-1}_{i_1 \cdots i_k} \sum_{v_1=1}^{k} p_{i_{v_1} j_2 \cdots j_k} p_{i_1 \cdots i_{v_1 - 1} j_1 i_{v_1 + 1} \cdots i_k}$$

$$= p^{k-2}_{i_1\dots i_k} \sum_{\nu_1=1}^{k} p_{i_{v_1} j_2\dots j_k} p_{i_1\dots i_k} b_1^{v_1}$$

$$= p^{k-2}_{i_1\dots i_k} \sum_{\nu_2=1}^{k} \sum_{\nu_1=1}^{k} p_{i_{v_1} i_{v_2} j_3\dots j_k} p_{i_1\dots i_{v_2-1} j_2 i_{v_2+1}\dots i_k} b_1^{v_1}$$

$$= p^{k-2}_{i_1\dots i_k} \sum_{\nu_2=1}^{k} \sum_{\nu_1=1}^{k} p_{i_{v_1} i_{v_2} j_3\dots j_k} b_2^{v_2} b_1^{v_1}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$= \sum_{\nu_k=1}^{k} \dots \sum_{\nu_2=1}^{k} \sum_{\nu_1=1}^{k} p_{i_{v_1} i_{v_2} \dots i_{v_k}} b_k^{v_k} \dots b_2^{v_2} b_1^{v_1} \quad .$$

Now, because $p_{i_{v_1}\dots i_{v_k}}$ is skew-symmetric in its suffixes, we have

$$p_{i_{v_1}\dots i_{v_k}} = p_{i_1\dots i_k}, \ -p_{i_1\dots i_k}, \quad \text{or} \quad 0$$

according as $v_1, \dots, v_k$ is an even permutation of $1, \dots, k$, an odd permutation, or the suffixes are not all distinct. Hence we have

$$p^k_{i_1\dots i_k} p_{j_1\dots j_k} = p_{i_1\dots i_k} \sum \epsilon \, b_1^{v_1} b_2^{v_2} \dots b_k^{v_k} \quad ,$$

where the summation extends over all permutations $v_1, \dots, v_k$ of the natural numbers $1, \dots, k$ and $\epsilon = +1$ or -1 according as the permutation is even or odd. Noting that this sum is just the required determinant, we see that this proves (5.2.2).

We are now ready to prove (5.2.1). To this end, let the $C(n,k)$ elements $(\ldots, p_{j_1 \ldots j_k}, \ldots)$ be given subject to the conditions of (5.2.1) and let $p_{i_1 \ldots i_k}$ be different from zero. Then set

$$(5.2.3) \qquad x^r_s = a\, p_{i_1 \ldots i_{r-1} s\, i_{r+1} \ldots i_k} \quad ,$$

where $r = 1, \ldots, k$, $s = 1, \ldots, n$, and a is a normalizing factor to be determined later; and consider the k-cell $((\theta, 1x^1, \ldots, kx^k))$ with initial vertex at the origin and vertices $rx^r = r\sum_{s=1}^{k} x^r_s e^s$, where x^r_s is defined by (5.2.3) and e^s is the s-th basis vector for $L_n(K)$. In particular, we have, for the i-th component $(1/k!)\underline{X}_{i_1 \ldots i_k}$ of $((\theta, 1x^1, \ldots, kx^k))$.

$$(1/k!)\underline{X}_{i_1 \ldots i_k} = \begin{vmatrix} 1 & 0 & . & 0 \\ 1 & x^1_{i_1} & . & x^1_{i_k} \\ . & . & . & . \\ 1 & x^k_{i_1} & . & x^k_{i_k} \end{vmatrix} \quad ;$$

and, since, by (5.2.3) and the skew-symmetry of the suffixes,

$$\begin{aligned} x^r_{i_s} &= a\, p_{i_1 \ldots i_{r-1}\, i_s\, i_{r+1} \ldots i_k} \\ &= a\, \delta^r_s\, p_{i_1 \ldots i_k} \quad , \end{aligned}$$

where δ^r_s is the Kronecker delta, it is clear that the above determinant reduces to a diagonal determinant and we have

$$(1/k!)\underline{X}_{i_1\ldots i_k} = (a^k\, p^{k-1}_{i_1\ldots i_k})\, p_{i_1\ldots i_k} \quad .$$

Obviously, we must choose a so that

$$a^k\, p^{k-1}_{i_1\ldots i_k} = 1 \quad , \tag{5.2.4}$$

which we can do since $p_{i_1\ldots i_k} \neq 0$ and K is algebraically closed. Hence if a be determined by (5.2.4), we have, for an arbitrary component $(1/k!)\underline{X}_{j_1\ldots j_k}$ of $((\theta,\ 1x^1,\ \ldots,\ kx^k))$, in view of (5.2.3),

$$(1/k!)\underline{X}_{j_1\ldots j_k} = \begin{vmatrix} 1 & 0 & \cdot & 0 \\ 1 & x^1_{j_1} & \cdot & x^1_{j_k} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} = \left| a\, b^r_s \right| \quad ,$$

where $b^r_s = p_{i_1\ldots i_{r-1}j_s i_{r+1}\ldots i_k}$. Factoring a out of each of the rows of the last determinant, applying (5.2.2), and using (5.2.4), we have

$$(1/k!)\underline{X}_{j_1\ldots j_k} = a^k \left| b^r_s \right| = a^k\, p^{k-1}_{i_1\ldots i_k} p_{j_1\ldots j_k} = p_{j_1\ldots j_k}.$$

This completes the proof of (5.2.1).

The next theorem follows as a consequence of the argument used in proving (5.2.4).

(5.2.5) <u>Theorem.</u> If $(\ldots, p_{j_1\ldots j_k}, \ldots)$ is any $C(n,k)$-tuple of elements of K satisfying the Grassmann quadratic relations and skew-symmetric in their suffixes and if $p_{i_1\ldots i_k}$ is any non-zero element of the set, then the k-cell $((\theta, 1x^1, \ldots, kx^k))$, where

$$x_s^r = a\, p_{i_1\ldots i_{r-1} j_s i_{r+1}\ldots i_k} \quad \text{with} \quad a^k = p^{1-k}_{i_1\ldots i_k},$$

has $(\ldots, p_{j_1\ldots j_k}, \ldots)$ for its components, provided K is algebraically closed and $1 < k \leqq n$.

<u>5.3. The Associated N-Space.</u> Let $N = C(n,k)$ and let $L_N(K)$ be the N-dimensional vector space over K defined as was $L_n(K)$ in (3.1.1). Let $L_N(K)$ be referred to the natural coordinate system or basis

$$\text{(5.3.1)} \qquad \begin{aligned} E^1 &= (1, 0, \ldots, 0), \\ E^2 &= (0, 1, \ldots, 0), \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ E^N &= (0, 0, \ldots, 1). \end{aligned}$$

We shall call $L_N(K)$, the N-space associated with $L_n(K)$, or simply the associated N-space. When K is the field of real (or complex) numbers we may sometimes denote $L_N(K)$ by R_N (or C_N). We define the inner product, the norm, and the distance functions in C_N (and R_N) respectively as in C_n (and R_n); see (4.1.1), ..., (4.1.14). We remark that C_n (or R_n) naturally implies C_N (or R_N). It is clear that the dimension of the associated N-space depends not only on n but also on our choice of k. Now we make the following remark in accordance with (1.1.2) and (1.1.3).

(5.3.2) Remark. The dimension of the associated N-space is 1 for $0 = k \leq n$ and 0 for $0 \leq n < k$. $L_0(K)$ contains only the null vector, i.e. consists of a single point (the origin).

Now let $((x^0, \ldots, x^k))$ be an arbitrary k-cell in $L_n(K)$ and consider the components

$$\xi_j = 1/k!\, \underline{X}_{j_1 \ldots j_k}, \qquad j = 1, \ldots, (N = C(n,k)),$$

where $j = C(n,k) - \sum_{v=1}^{k} C(n-j_v, k-v+1)$ as in (1.5.1), of the k-vector

$$\begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix}$$

determined by it. It follows from (3.3.4) that the components are elements of K and, consequently, the k-vector determined by $((x^0, \ldots, x^k))$ is a well determined vector in $L_N(K)$. Thus we see that the correspondence

$$(5.3.3) \qquad ((x^0, \ldots, x^k)) \rightarrow \begin{pmatrix} x^0 \\ \vdots \\ x^k \end{pmatrix}$$

relates to each k-cell of $L_n(K)$ a unique point or vector in $L_N(K)$, and, therefore, effects a mapping of the k-cells of $L_n(K)$ in $L_N(K)$. It may happen that the mapping effected by (5.3.3) fails to fill out $L_N(K)$; indeed this is exactly the case whenever $1 < k < n-1$. Hence we shall term k-vectors those vectors in $L_N(K)$ which are images of k-cells of $L_n(K)$ under (5.3.3) in order to distinguish them.

We remark further that under the definitions (3.4.1) and (3.4.2) the sum of two k-vectors is a vector in $L_N(K)$ (not necessarily a k-vector) and the product of a k-vector by an element of the ground field is a vector in $L_N(K)$, since these definitions are, in fact, the same as those in (3.1.1), with n replaced by N. Moreover, it is apparent upon comparing (4.1.1), (4.1.2), and (4.1.10) (with n replaced by N) respectively with (4.1.15), (4.1.16) and (4.1.17) that the values for the inner product, the norm, and the distance of k-cells over C_n (or R_n) are identical with those of the corresponding functions of their k-vectors in C_N (or R_N). Thus we see that (5.3.3) preserves trivially the inner product, norm, and distance as defined in C_n (or R_n) and C_N (or R_N).

The following theorem is an immediate consequence of

(3.3.11) and the foregoing remarks.

(5.3.4) Theorem. The correspondence (5.3.3) determines a many-to-one mapping of the set of all k-cells over the base space $L_n(K)$ in the associated N-space $L_N(K)$; when K is the field of complex (or real) numbers (5.3.3) preserves the inner product, the norm, and the distance functions.

Since only those vectors in the associated N-space which are images of k-cells under (5.3.3) are to be called k-vectors, it follows from (5.2.1) that the next theorem is true.

(5.3.5) Theorem. Every k-vector in the associated N-space is the image of at least one k-cell with initial vertex at the origin in the base space.

Now we make the following remark for future reference.

(5.3.6) Remark. A k-vector is a proper k-vector if, and only if, it is the image of a proper k-cell.

Next we note that the k-vectors respectively determined by the k-cells

$$((\theta, 1\ e^{i_1}, 2\ e^{i_2}, \ldots, k\ e^{i_k}))$$

are the vectors (5.3.1). Hence it is true that

(5.3.7) Theorem. The natural basis for the associated

N-space consists of k-vectors.

Now it is obvious that whenever a $C(n,k)$-tuple of elements of K satisfy the Grassmann quadratic relations then so does every $C(n,k)$-tuple whose components are proportional to those of the first. Hence we see that it is true, by (5.2.1), that

(5.3.8) Theorem. For every k-vector ξ in the associated N-space and every element a in K the corresponding vector $a\xi$ is again a k-vector, provided K is algebraically closed and $0 < k$.

From the identity

$$\begin{vmatrix} x^1_{j_1}+y_{j_1} & \cdot & x^1_{j_k}+y_{j_k} \\ x^2_{j_1} & \cdot & x^2_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} = \begin{vmatrix} x^1_{j_1} & \cdot & x^1_{j_k} \\ x^2_{j_1} & \cdot & x^2_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix} + \begin{vmatrix} y_{j_1} & \cdot & y_{j_k} \\ x^2_{j_1} & \cdot & x^2_{j_k} \\ \cdot & \cdot & \cdot \\ x^k_{j_1} & \cdot & x^k_{j_k} \end{vmatrix},$$

we infer that

$$\begin{pmatrix} \theta \\ x^1+y \\ x^2 \\ \vdots \\ x^k \end{pmatrix} = \begin{pmatrix} \theta \\ x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix} + \begin{pmatrix} \theta \\ y \\ x^2 \\ \vdots \\ x^k \end{pmatrix}$$

and, consequently, that under certain conditions the sum of two k-vectors is again a k-vector. We now proceed to show that there are vectors in the associated N-space which are the sum of two k-vectors but are not themselves k-vectors. To this end, let X be any proper k-vector in the associated N-space and let X_j be any non-zero component of X. We know by (5.3.7) that E^j, see (5.3.1), is also a proper k-vector. Hence the vector

$$Z = X + E^j$$

is a vector in $L_N(K)$, which is the sum of two k-vectors. Now, for the components Z_i of Z, we have

$$Z_i = X_i \quad \text{for} \quad i \neq j \quad \text{and} \quad Z_j = X_j + 1.$$

Now, by (5.1.2) and (5.3.6), Z is a k-vector if, and only if, for $1 < k < n-1$,

$$Z_i Z_j = Z_{i_1 \ldots i_k} Z_{j_1 \ldots j_k}$$

$$= \sum_{s=1}^{k} Z_{i_1 \ldots i_{r-1} j_s i_{r+1} \ldots i_k} Z_{j_1 \ldots j_{s-1} i_r j_{s+1} \ldots j_k}$$

$$= \sum_{s=1}^{k} X_{i_1 \ldots i_{r-1} j_s i_{r+1} \ldots i_k} X_{j_1 \ldots j_{s-1} i_r j_{s+1} \ldots j_k}$$

$$= X_i X_j \quad .$$

But $$Z_i Z_j = X_i (X_j + 1)$$

$$= X_i X_j + X_i \quad .$$

Hence it is necessary that $X_i = 0$ for all i different from j in order that Z be a k-vector. But we know that there are many proper k-vectors for which this is not true. We sum up in the next theorem.

(5.3.9) Theorem. The set of k-vectors in the associated N-space is closed under multiplication by scalars when K is algebraically closed and $0 < k$, but not under vector addition when $1 < k < n-1$.

One immediate conclusion to be drawn from (5.3.9) is the following remark.

(5.3.10) Remark. The totality of k-vectors forms a proper subset of the vectors of the associated N-space when $1 < k < n-1$.

5.4. Equivalence Classes. We have seen, (5.3.4), that the correspondence (5.3.3) determines a many-to-one mapping of k-cells on a subset, (5.3.10), of the elements of the associated N-space. In order that the so determined mapping be 1-1 we make the following definition.

(5.4.1) Definition. Two k-cells $((x^0,\ldots,x^k))$, $((y^0,\ldots,y^k))$ in $L_n(K)$ which determine the same k-vector in $L_N(K)$ shall be termed equivalent. We write

$$((x^0,\ldots,x^k)) \simeq ((y^0,\ldots,y^k)).$$

Clearly, $((x^0,\ldots,x^k)) \simeq ((y^0,\ldots,y^k))$ implies $((y^0,\ldots,y^k)) \simeq ((x^0,\ldots,x^k))$; hence the relation $\simeq$ is symmetric. It is obviously reflexive, i.e., $((x^0,\ldots,x^k)) \simeq ((x^0,\ldots,x^k))$. Moreover, $((x^0,\ldots,x^k)) \simeq ((y^0,\ldots,y^k))$ and $((y^0,\ldots,y^k)) \simeq ((z^0,\ldots,z^k))$ implies $((x^0,\ldots,x^k)) \simeq ((z^0,\ldots,z^k))$; hence it is transitive. Therefore the next remark holds.

(5.4.2) <u>Remark.</u> The relation $\simeq$ is a true equivalence relation.

It follows from (3.3.11) that for every k-cell $((x^0,\ldots,x^k))$ in $L_n(K)$, it is true that

$$((x^0,\ldots,x^k)) \simeq ((\theta,\ x^1-x^0,\ \ldots,\ x^k-x^0)).$$

Hence we have the next theorem.

(5.4.3) <u>Theorem.</u> Every k-cell is equivalent to at least one k-cell with initial vertex at the origin.

Before proceeding further we should make the following obvious remark, which follows from (5.4.1) and (5.3.6).

(5.4.4) <u>Remark.</u> Proper k-cells are equivalent only to proper k-cells; all null k-cells are equivalent.

All k-cells which are equivalent, under (5.4.1), to a given k-cell $((x^0,\ldots,x^k))$ constitute the equivalence class $\{((x^0,\ldots,x^k))\}$. Evidently each member of the class serves to determine the class. Since every k-cell, by (5.4.3), is equivalent to at least one k-cell with initial vertex at the origin, every k-cell lies in an equivalence class determined by some k-cell with initial vertex at the origin. Hence we see, in view of (5.3.5), that the correspondence

(5.4.5) $$\{((\theta,x^1,\ldots,x^k))\} \leftrightarrow \begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^k \end{pmatrix}$$

is 1-1, and biunique. Hence we have the following theorem.

(5.4.6) <u>Theorem.</u> The correspondence (5.4.5) determines a 1-1, biunique mapping of the set whose elements are the equivalence classes under (5.4.1) on the set of k-vectors in the associated N-space.

In particular, it follows from the latter half of (5.3.4) that the next theorem is true.

(5.4.7) <u>Theorem.</u> The set whose elements are the equivalence classes over C_n (or R_n) and the set of k-vectors in C_N (or R_N) are isomorphic under (5.4.5); the isomorphism preserves the inner product, the norm, and the distance functions.

Now we inquire into the relations connecting k-cells which fall into the same equivalence class. Since all null k-cells are equivalent, by (5.4.4), they all fall in the same equivalence class, which we shall call the null equivalence class. Hence we have

(5.4.8) Theorem. The set of all null k-cells constitutes the null equivalence class.

Now let $((x^0,\dots,x^k))$ be any proper k-cell in $L_n(K)$. It determines the equivalence class $\{((x^0,\dots,x^k))\}$. Let $((y^0,\dots,y^k))$ be any k-cell in $\{((x^0,\dots,x^k))\}$. Then, by (5.4.4), $((y^0,\dots,y^k))$ is also a proper k-cell. Since both $((x^0,\dots,x^k))$, $((y^0,\dots,y^k))$ are proper k-cells it follows, by (3.1.6), that their respective matrices of edges at the vertices X, Y both have rank k. Furthermore, since they are equivalent, their corresponding components are equal, i.e., by (3.3.5),

$$(1/k!)\ Y^{(k)} = (1/k!)\ X^{(k)} \quad .$$

Thus it is apparent that in order that two proper k-cells belong to the same equivalence class it is necessary and sufficient that the k-th compounds of their matrices of edges at the initial vertices be equal. Since X, Y have rank k, this is true, by (2) of (1.4.11), if, and only if, there exist non-singular matrices C of order k, D of order n, T and S of order k such that

$$C\,Y\,D = [T\ 0] \quad \text{and} \quad C\,X\,D = [S\ 0]\ ,$$

where $|T| = |S|$. Hence we have the following theorem.

(5.4.8) <u>Theorem.</u> Two proper k-cells $((x^0,\dots,x^k))$, $((y^0,\dots,y^k))$ whose matrices of edges at their initial vertices are respectively X, Y belong to the same equivalence class if, and only if, there exist non-singular matrices C, T, S of order k and D of order n such that

$$C\,Y\,D = [T\ 0],\quad C\,S\,D = [S\ 0], \quad \text{and} \quad |T| = |S|\ .$$

Now if $((x^0,\dots,x^k))$, $((y^0,\dots,y^k))$ be any pair of equivalent k-cells, then, by (5.4.8), it is necessary that

$$C\,Y\,D = [T\ 0] \quad \text{and} \quad C\,X\,D = [S\ 0]\ .$$

Hence it is necessary that

$$T^{-1}\,C\,Y\,D = [I_k\ 0] \quad \text{and} \quad S^{-1}\,C\,X\,D = [I_k\ 0]$$

and, consequently, that

$$T^{-1}\,C\,Y\,D = S^{-1}\,C\,X\,D\ .$$

From this, it follows that it is necessary that

$$Y = C^{-1}\,T\,S^{-1}\,C\,X\ .$$

Setting

$$A = C^{-1}\ T\ S^{-1}\ C,$$

we have

$$|A| = \frac{1}{|C|}\ |T|\ \frac{1}{|S|}\ |C|$$

$$= 1,$$

since, by (5.4.8), $|T| = |S|$. Therefore, we see that it is necessary that there exist a $k \times k$ unimodular matrix A such that $Y = A\,X$. Conversely, if $Y = A\,X$ and $|A| = 1$, we have

$$(1/k!)\ Y^{(k)} = (1/k!)\ |A|\ X^{(k)} = (1/k!)\ X^{(k)}\ .$$

Hence we have proved the next theorem.

(5.4.9) <u>Theorem.</u> A necessary and sufficient condition that two proper k-cells belong to the same equivalence class is that their matrices of edges at their initial vertices X, Y be related as follows:

$$Y = A\,X \quad \text{where} \quad |A| = 1\ .$$

<u>5.5. The Grassmann Quadrics.</u> In this section we consider the relation that the set of k-vectors in the

associated N-space bears to the entire space $L_N(K)$.

When $k > n \geqq 0$, we know, by (3.1.7), that all k-cells are null k-cells, and, hence, by (5.4.8), constitute the null equivalence class. The associated N-space is then, by (5.3.2), $L_0(K)$ containing the origin only. In this case the set of k-vectors consists of only the null-vector, the only vector in $L_0(K)$. Hence we have the remark.

(5.5.1) Remark. When $0 \leqq n < k$, the set of k vectors consists of only the null vector which in turn constitutes the entire associated N-space $L_0(K)$.

Since we have agreed to ascribe the single component unity to each 0-cell and since, by (5.3.2), the associated N-space is $L_1(K)$ when $0 = k \leqq n$, we have the next remark.

(5.5.2) Remark. When $0 = k \leqq n$, the set of k-vectors consists of the unit vector in the associated N-space $L_1(K)$.

It follows from the remark (3.4.10) and (5.4.3) that the next remark is true.

(5.5.3) Remark. When $1 = k \leqq n$, the set of k-vectors constitutes the entire associated N-space $L_N(K) = L_n(K)$.

It follows from (5.4.3) that we need only consider n-cells with initial vertex at the origin when attempting to find the set of n-vectors. Now such an n-cell has but one component, namely

$$(1/n!) \begin{vmatrix} x_1^1 & \cdot & x_n^1 \\ \cdot & \cdot & \cdot \\ x_1^n & \cdot & x_n^n \end{vmatrix} .$$

It is easy to see that the set of all such n-cells will determine all 1-vectors in $L_1^*(K)$. Hence we have

(5.5.4) <u>Remark.</u> When $1 \leqq k = n$, the set of k-vectors constitutes the associated N-space, namely, $L_1(K)$.

It follows from (5.1.4) and (5.2.5), that the next remark is true.

(5.5.5) <u>Remark.</u> When $1 \leqq k = n-1$, the set of k-vectors constitutes the associated N-space $L_N(K) = L_n(K)$.

Referring to (5.1.2), we see that when $2 = k < n-1$, the components of a k-cell must satisfy $C(n, 4)$ distinct relations of the form

$$X_{i_1 i_2} X_{j_1 j_2} = X_{j_1 i_2} X_{i_1 j_2} + X_{j_2 i_2} X_{j_1 i_1} ;$$

for there will be just as many distinct relations as there are distinct quadruples i_1, i_2, j_1, j_2 of integers in the set 1, 2, ..., n. Furthermore, if $1 < k = n-2$, there will be as many distinct quadratic relations of the form

$$X_{i_1\cdots i_{n-2}} X_{j_1\cdots j_{n-2}}$$

$$= \sum_{s=1}^{n-2} X_{i_1\cdots i_{r-1} j_s i_{r+1}\cdots i_{n-2}} X_{j_1\cdots j_{s-1} i_r j_{s+1}\cdots j_{n-2}}$$

as there are ways of omitting quadruples $i_{n-1} i_n j_{n-1} j_n$ from the set 1, 2, ..., n, which is again $C(n,4)$ in number. Since each quadratic relation can be written as a symmetrical quadratic form equated to zero, we have the following theorem.

(5.5.6) Theorem. When $n \geq 4$ and $k = 2$ or $k = n-2$, the set of k-vectors coincides with the points common to $C(n,4)$ quadric hypersurfaces in the associated N-space, provided K is algebraically closed.

More generally, we have the following theorem.

(5.5.7) Theorem. When $n > 4$ and $1 < k < n-1$, the set of k-vectors coincides with the vectors determining the intersection of the set of quadric hypersurfaces determined by the Grassmann quadratic relations, provided K is algebraically closed.

5.6. An Example. In the simplest case for which the Grassmann quadratic relations yield a quadric hypersurface ($k = 2$, $n = 4$) we have, for an arbitrary 2-cell with initial vertex at the origin,

$$\begin{pmatrix} \theta \\ x^1 \\ x^2 \end{pmatrix} = \tfrac{1}{2}\,(X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}),$$

where

$$X_{ij} = \begin{vmatrix} x_i^1 & x_j^1 \\ x_i^2 & x_j^2 \end{vmatrix} .$$

From the identity

$$\begin{vmatrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{vmatrix} = 0,$$

one obtains, by a Laplace expansion in terms of the first two rows, the equivalent identity

(5.6.1)

$$X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} + X_{23}X_{14} - X_{24}X_{13} + X_{34}X_{12} = 0 .$$

This identity which every 2-vector in $L_4(K)$ must satisfy is usually written in the shorter form, which agrees with (5.1.2),

$$X_{12}\,X_{34} + X_{13}\,X_{42} + X_{14}\,X_{23} = 0.$$

see, for example, [3] .

Under the correspondence

(5.6.2)
$$X_{12} \to \xi_1,\ X_{13} \to \xi_2,\ X_{14} \to \xi_3,$$
$$X_{23} \to \xi_4,\ X_{24} \to \xi_5,\ X_{34} \to \xi_6,$$

the components of the 2-vector may be regarded as the components of the vector $(\xi_1, \ldots, \xi_6)$ in $L_6(K)$, which, by (5.6.1), must satisfy the condition

$$(5.6.3) \quad \xi_1\xi_6 - \xi_2\xi_5 + \xi_3\xi_4 + \xi_4\xi_3 - \xi_5\xi_2 + \xi_6\xi_1 = 0.$$

Conversely, we know, by (5.2.1), that any vector $(\xi_1, \ldots, \xi_6)$ satisfying (5.6.3) is a 2-vector. Hence we see that those, and only those, vectors in $L_6(K)$ which satisfy (5.6.3) are 2-vectors. Now (5.6.3) can be written in the equivalent matrix form

$$(5.6.4) \qquad \xi Q \xi^* = 0,$$

where $\xi = [\xi_1 \ \xi_2 \ \xi_3 \ \xi_4 \ \xi_5 \ \xi_6]$

and
$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now if we make the transformation of coordinates

$$\eta = \xi P ,$$

where

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we have $\xi = \eta P^{-1}$, and (5.6.4) becomes

$$\eta P^{-1} Q P^{*-1} \eta^{*} = 0 ,$$

which reduces to

$$\eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_4^2 - \eta_5^2 - \eta_6^2 = 0 .$$

This shows that (5.6.4) is an indefinite quadratic form of rank 6 and signature zero, whenever the base space is R_4; of course, therefore, the Grassmann quadratic relation is equivalent to (5.6.4).

<u>5.7. The Grassmann Matrices.</u> Let $\{((\theta, x^1, \ldots, x^k))\}$ be an arbitrary equivalence class of k-cells in $L_n(K)$. Then, under the correspondence

$$\begin{matrix} X_{1\ldots k}, & \ldots, & X_{i_1 \ldots i_k}, & \ldots, & X_{(n-k+1)\ldots n} \\ \updownarrow & & \updownarrow & & \updownarrow \\ X_1 & , \ldots, & X_i & , \ldots, & X_N \end{matrix} ,$$

where the $X_{i_1 \cdots i_k}$ are the components of $((\theta, x^1, \ldots, x^k))$ taken in lexicographical order, the Grassmann quadratic relations of (5.1.2) can be expressed in terms of the components of the k-vector

$$X = (X_1, \ldots, X_N), \qquad N = C(n,k),$$

determined by the equivalence class $\{((\theta, x^1, \ldots, x^k))\}$, namely,

$$X_i X_j = \sum_{s=1}^{k} \epsilon_{(i_r)_s} \epsilon_{(j_s)_r} X_{(i_r)_s} X_{(j_s)_r} ,$$

where $\epsilon_{(i_r)_s} X_{(i_r)_s} = X_{i_1 \cdots i_{r-1} j_s i_{r+1} \cdots i_k}$,

$$\epsilon_{(j_s)_r} X_{(j_s)_r} = X_{j_1 \cdots j_{s-1} i_r j_{s+1} \cdots j_k} ,$$

$\epsilon_{(i_r)_s} = +1$ or -1 according as the sequence

$i_1, \ldots, i_{r-1}, j_s, i_{r+1}, \ldots, i_k$ is an even or an odd permutation of the same set of integers taken in natural order when they are all distinct and $\epsilon_{(i_r)_s} = 0$ otherwise, and $\epsilon_{(j_s)_r} = +1$ or -1 according as

$$j_1, \ldots, j_{s-1}, i_r, j_{s+1}, \ldots, j_k$$

is an even or odd permutation of the same set taken in natural order when they are all distinct and $\epsilon_{(j_s)_r} = 0$ otherwise. These relations can be symmetrized by writing them in the equivalent form

$$X_i X_j - \sum_{s=1}^{\kappa} \epsilon_{(i_r)_s} \epsilon_{(j_s)_r} X_{(i_r)_s} X_{(j_s)_r}$$

$$- \sum_{s=1}^{\kappa} \epsilon_{(j_s)_r} \epsilon_{(i_r)_s} X_{(j_s)_r} X_{(i_r)_s} + X_j X_i = 0 .$$

These relations are simply symmetric quadratic forms

$$(5.7.1) \qquad \sum_{u,v=1}^{N} X_u \begin{bmatrix} Q^{n,k}_{rij} \end{bmatrix}^u_v X_v = 0 ,$$

where

$$\begin{bmatrix} Q^{n,k}_{rij} \end{bmatrix}^i_j = \begin{bmatrix} Q^{n,k}_{rij} \end{bmatrix}^j_i = 1 ,$$

$$\begin{bmatrix} Q^{n,k}_{rij} \end{bmatrix}^{(i_r)_s}_{(j_s)_r} = - \epsilon_{(i_r)_s} \epsilon_{(j_s)_r} = - \epsilon_{(j_s)_r} \epsilon_{(i_r)_s} = \begin{bmatrix} Q^{n,k}_{rij} \end{bmatrix}^{(j_s)_r}_{(i_r)_s} ,$$

and $\begin{bmatrix} Q^{n,k}_{rij} \end{bmatrix}^u_v = 0$ otherwise. These relations can therefore be written in matrix notation as follows:

$$(5.7.2) \qquad X \begin{bmatrix} Q^{n,k}_{rij} \end{bmatrix} X' = 0 ,$$

where X is the row matrix $[X_1 \dots X_N]$, X' is the column matrix obtained by transposing X, 0 is the zero matrix of order 1, and $\begin{bmatrix} Q^{n,k}_{rij} \end{bmatrix}$ is the symmetric $N \times N$ matrix of the quadratic form. In this manner we associate k symmetric $N \times N$ matrices with each pair of components of the k-vector X. The kN^2 matrices which may be obtained in this manner will be called the <u>special Grassmann matrices</u> for the associated N-space, or simply the special Grassmann matrices. Furthermore, whenever a vector X in the associated N-space, regarded as a row matrix, satisfies (5.7.2), for any matrix $Q^{n,k}_{rij}$, we shall say that the vector X annihilates the matrix $Q^{n,k}_{rij}$. Using this terminology, we make the following remark.

(5.7.3) <u>Remark.</u> The k-vectors are those, and only those, vectors which annihilate the special Grassmann matrices, $1 < k < n-1$.

The next remark follows from (5.1.3).

(5.7.4) <u>Remark.</u> Whenever the sets $\{i_1 \dots i_k\}$ and $\{j_1 \dots j_k\}$ differ by less than four integers the corresponding special Grassmann matrices are annihilated by every vector in the associated N-space of $L_n(K)$.

If we define the special Grassmann matrices to be the $N \times N$ zero matrices in the cases $k = 1$ and $k = n-1$, we can remove the restriction $1 < k < n-1$ in (5.7.3).

Now we term the set $\{Q^{n,k}\}$ of all $N \times N$ matrices obtainable as linear combinations

$$Q^{n,k} = \sum_{r=1}^{k} \sum_{i,j=1}^{N} a_{rij} \left[Q^{n,k}_{rij} \right] \tag{5.7.5}$$

of the special Grassmann matrices, where the coefficients a_{rij} are elements of the ground field K, the set of Grassmann matrices for the associated N-space, or simply the Grassmann matrices.

We are ready for the main theorem of this section.

(5.7.6) <u>Theorem.</u> A vector in the associated N-space is a k-vector if, and only if, it annihilates the set $\{Q^{n,k}\}$ of Grassmann matrices.

We prove the condition necessary first. Suppose the vector X does not annihilate $\{Q^{n,k}\}$. Then there is a Grassmann matrix Q which X does not annihilate. This means that there is a linear combination of the special Grassmann matrices such that

$$0 \neq X \left[\sum_{h,i,j} a_{rij} \left[Q^{n,k}_{rij} \right] \right] X'$$

$$= \sum_{h,i,j} a_{rij} \, X \left[Q^{n,k}_{rij} \right] X' \quad .$$

Hence there must be at least one term of the last sum for which

$$X \left[Q^{n,k}_{rij} \right] X' \neq 0 \quad .$$

Therefore, by (5.7.3), X is not a k-vector. Conversely, suppose X annihilates $\{Q^{n,k}\}$. Then, for every set of kN^2 elements of K, we have

$$0 = X\left[\sum_{\lambda,i,j} a_{rij}\left[Q^{n,k}_{rij}\right]\right]X'$$
$$= \sum_{\lambda,i,j} a_{rij}\, X\left[Q^{n,k}_{rij}\right]X' \quad .$$

We need merely choose kN^2 sets of coefficients in such a way that we obtain kN^2 linearly independent equations in

$$X\left[Q_{111}\right]X', \ldots, X\left[Q_{k,N-1,N}\right]X'$$

to show that

$$X\left[Q^{n,k}_{rij}\right]X' = 0$$

for all rij . This can be done in an infinite number of ways. Hence X annihilates the special Grassmann matrices and is, by (5.7.3), a k-vector. This completes the proof of (5.7.6).

It is evident from the discussion leading up to (5.7.2) that the next remarks are true.

(5.7.7) <u>Remark.</u> If Q is a special Grassmann matrix then so is $-Q$.

(5.7.8) <u>Remark.</u> The matrices of the set $\left\{Q^{n,k}_{rij}\right\}$ of special Grassmann matrices are not all distinct.

<u>5.8. Duality Considerations.</u> Let $\left\{Q^{n,k}_{rij}\right\}$ denote the set of special Grassmann matrices for k-vectors and $\left\{Q^{n,n-k}_{rij}\right\}$ denote the set of special Grassmann matrices for

(n-k)-vectors when the base space is $L_n(K)$ and k, $1 < k < n-1$, is arbitrary but fixed. Then, under the correspondence described in Section 5.7, if $Q_{rij}^{n,k}$ be any element of the set $\{Q_{rij}^{n,k}\}$, there exists a unique determining Grassmann quadratic relation, namely,

$$X_{i_1 \cdots i_k} X_{j_1 \cdots j_k}$$

$$= \sum_{s=1}^{k} X_{i_1 \cdots i_{r-1} j_s i_{r+1} \cdots i_k} X_{j_1 \cdots j_{s-1} i_r j_{s+1} \cdots j_k} ,$$

where $1 \leqq i_1 < \cdots < i_k \leqq n$, $1 \leqq j_1 < \cdots < j_k \leqq n$, $(i_1, \ldots, i_k)$ is the i-th term and $(j_1, \ldots, j_k)$ is the j-th term of the sequence of k-tuples of the first n natural numbers under lexicographical ordering. Thus to each element of the set $\{Q_{rij}^{n,k}\}$ there corresponds a unique Grassmann quadratic relation. Now we make correspond to each Grassmann quadratic relation for k-vectors a unique Grassmann quadratic relation for (n-k)-vectors in the following manner. Let the integers which remain after $i_1, \ldots, i_k$ (or $j_1, \ldots, j_k$) have been deleted from the first n natural numbers be denoted by $i'_1, \ldots, i'_{n-k}$ (or $j'_1, \ldots, j'_{n-k}$), where

$1 \leqq i'_1 < \cdots < i'_{n-k} \leqq n$, $1 \leqq j'_1 < \cdots < j'_{n-k} \leqq n$, $(i'_1, \ldots, i'_{n-k})$

is the i'-th term and $(j'_1, \ldots, j'_{n-k})$ is the j'-th term of the sequence of (n-k)-tuples of the first n natural numbers under lexicographical ordering and let the indeterminates $Y_{u'_1 \cdots u'_{n-k}}$ be skew-symmetric in their suffixes. Then either i_r is an integer of the set $\{j_1, \ldots, j_k\}$ or i_r is an integer of the set $\{j'_1, \ldots, j'_{n-k}\}$, say $i_r = j'_{s'}$.

In the first case the Grassmann quadratic relation corresponding to $Q^{n,k}_{rij}$ becomes simply

$$X_{i_1 \ldots i_k} X_{j_1 \ldots j_k} = X_{i_1 \ldots i_k} X_{j_1 \ldots j_k} ,$$

because the indeterminates $X_{v_1 \ldots v_k}$ are skew-symmetric in their suffixes, and we see that $Q^{n,k}_{rij}$ is the zero matrix of order $C(n,k)$. In this case the corresponding Grassmann quadratic relation for (n-k)-vectors is to be taken as

$$Y_{i'_1 \ldots i'_{n-k}} Y_{j'_1 \ldots j'_{n-k}} = Y_{i'_1 \ldots i'_{n-k}} Y_{j'_1 \ldots j'_{n-k}} ,$$

which, in turn, determines the zero matrix of the set $\{Q^{n,n-k}_{rij}\}$.

In the second case, the corresponding Grassmann quadratic relation is to be taken as

$$Y_{i'_1 \ldots i'_{n-k}} Y_{j'_1 \ldots j'_{n-k}}$$

$$= \sum_{r'=1}^{n-k} Y_{i'_1 \ldots i'_{r'-1} j'_{s'} i'_{r'+1} \ldots i'_{n-k}} Y_{j'_1 \ldots j'_{s'-1} i'_{r'} j'_{s'+1} \ldots j'_{n-k}} ,$$

where $j'_{s'} = i_r$, as noted before.

Thus we have established a correspondence which relates to each distinct element of the set $\{Q^{n,k}_{rij}\}$ a distinct element of the set $\{Q^{n,n-k}_{rij}\}$--it is evident from the discussion at the beginning of Section 5.7 that each set contains kN^2 elements not all of which are distinct.

Since the above process is clearly reversible, we can make the correspondence go both ways to obtain

(5.8.1) $$Q^{n,k}_{rij} \leftrightarrow Q^{n,n-k}_{s'i'j'} \quad , \quad i_r = j'_{s'} \quad ,$$

for non-zero special Grassmann matrices of the sets $\{Q^{n,k}_{rij}\}$ and $\{Q^{n,n-k}_{rij}\}$, where, in view of the discussion of Section 1.6,

$$i + i' = j + j' = N + 1, \qquad N = C(n,k) = C(n,n-k),$$

and $\quad 0_{C(n,k)} \leftrightarrow 0_{C(n,n-k)}, \quad$ otherwise.

We term the correspondents under (5.8.1) duals; and state the next theorem.

(5.8.2) Theorem. The correspondence (5.8.1) relates to each special Grassmann matrix for k-vectors a dual special Grassmann matrix for (n-k)-vectors, and conversely.

Now let us examine the matrices paired under (5.8.1). To this end, let $Q^{n,k}_{rij}$ be an arbitrary non-zero special Grassmann matrix. Its determining Grassmann quadratic relation can be put in the equivalent form, wherein $j_0 = i_r$,

$$\sum_{s=0}^{k}(-)^{r+s}X_{j_s i_1 \cdots i_{r-1} i_{r+1} \cdots i_k} X_{j_0 j_1 \cdots j_{s-1} j_{s+1} \cdots j_k} = 0 \text{ .}$$

If $\{h_1, \ldots, h_m\}$ be the set of integers, arranged in their natural order, which are common to the sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_k\}$, we know, since $Q^{n,k}_{rij}$ is a non-zero matrix, that i_r is not in the set $\{j_1, \ldots, j_k\}$ and therefore not in the set $\{h_1, \ldots, h_m\}$, which shows that $\{i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_k\}$ contains $\{h_1, \ldots, h_m\}$. But j_s runs over the entire set $\{j_1, \ldots, j_k\}$ and, in so doing, must run over the set $\{h_1, \ldots, h_m\}$. Now, because $X_{j_s i_1 \cdots i_{r-1} i_{r+1} \cdots i_k}$ is skew-symmetric in its suffixes, those terms of the quadratic relation for which j_s is an integer of the set $\{h_1, \ldots, h_m\}$ must vanish. Thus we see that the above relation reduces to the simpler form

$$\sum_{\nu=0}^{k-m}(-)^{r+s_\nu} X_{j_{s_\nu} i_1 \cdots i_{r-1} i_{r+1} \cdots i_k} X_{j_0 j_1 \cdots j_{s_\nu -1} j_{s_\nu +1} \cdots j_k} = 0,$$

where $s_0 = 0$, $j_0 = i_r$, and the set $\{j_{s_1}, \ldots, j_{s_{k-m}}\}$ consists of those integers, arranged in their natural order, of the set $\{j_1, \ldots, j_k\}$ which are not in the set $\{i_1, \ldots, i_k\}$.

By an entirely similar argument, we deduce from the determining Grassmann quadratic relation

$$\sum_{\lambda'=0}^{n-k}(-)^{r'+s'} Y_{i'_0 i'_1 \cdots i'_{r'-1} i'_{r'+1} \cdots i'_{n-k}} Y_{i'_{r'} j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}} = 0$$

for the dual special Grassmann matrix $Q^{n,n-k}_{s'i'j'}$, where

$$i'_0 = j'_{s'} = i_r \quad \text{and} \quad i + i' = j + j' = N + 1 ,$$

the simpler equivalent relation

$$\sum_{\nu=0}^{n-k-m'}(-)^{r'_\nu+s'} Y_{i'_0 i'_1 \cdots i'_{r'_\nu-1} i'_{r'_\nu+1} \cdots i'_{n-k}} Y_{i'_{r'_\nu} j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}} = 0,$$

where, if $\{h'_1, \ldots, h'_{m'}\}$ be the set of integers, arranged in their natural order, common to the sets $\{i'_1, \ldots, i'_{n-k}\}$ and $\{j'_1, \ldots, j'_{n-k}\}$, the set $\{i'_{r'_1}, \ldots, i'_{r'_{n-k-m'}}\}$ consists of those integers, arranged in natural order, of the set $\{i'_1, \ldots, i'_{n-k}\}$ which are not in $\{j'_1, \ldots, j'_{n-k}\}$.

Now, since m integers, $h_1 < \ldots < h_m$, of the set $\{i_1, \ldots, i_k\}$ are also integers of the set $\{j_1, \ldots, j_k\}$, it is evident that there are $2k-m$ integers in their set theoretical sum $\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_k\}$. Furthermore there are m' integers, $h'_1 < \ldots < h'_{m'}$, in the intersection $\{i'_1, \ldots, i'_{n-k}\} \cap \{j'_1, \ldots, j'_{n-k}\}$ of their complements in $\{1, \ldots, n\}$. But it is easy to see that the set theoretical sum of $\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_k\}$ and

$\{i'_1, \ldots, i'_{n-k}\} \cap \{j'_1, \ldots, j'_{n-k}\}$ is the set $\{1, \ldots, n\}$ since $\{i'_1, \ldots, i'_{n-k}\}$ and $\{j'_1, \ldots, j'_{n-k}\}$ are respectively the complements of $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_k\}$ in $\{1, \ldots, n\}$. Therefore we have $2k-m+m' = n$, that is

$$m' = n - 2k + m .$$

Using this, we summarize the foregoing discussion in (5.8.3).

(5.8.3) <u>Theorem</u>. If $Q^{n,k}_{rij}$ and $Q^{n,n-k}_{s'i'j'}$ be any pair of dual special Grassmann matrices, then their determining Grassmann quadratic relations have respectively the forms

$$\sum_{\nu=0}^{k-m}(-)^{r+s_\nu} X_{j_{s_\nu} i_1 \cdots i_{r-1} i_{r+1} \cdots i_k} X_{j_0 j_1 \cdots j_{s_\nu - 1} j_{s_\nu + 1} \cdots j_k} = 0$$

and

$$\sum_{\nu=0}^{k-m}(-)^{r'_\nu + s'} Y_{i'_0 i'_1 \cdots i'_{r'_\nu - 1} i'_{r'_\nu + 1} \cdots i'_{n-k}} Y_{i'_{r'_\nu} j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}} = 0,$$

where $s_0 = r'_0 = 0$, $j_0 = i'_0 = j'_{s'} = i_r$, $i+i' = j+j' = N + 1$, $N = C(n,k)$, $(i_1, \ldots, i_k)$ is the i-th term and $(j_1, \ldots, j_k)$ is the j-th term of the sequence of k-tuples of the first n natural numbers taken in lexicographical order, $(i'_1, \ldots, i'_{n-k})$ is the i'-th term and $(j'_1, \ldots, j'_{n-k})$ is the j'-th term of the sequence of $(n-k)$-tuples of the first n natural numbers taken in lexicographical order, $i'_1 < \ldots < i'_{n-k}$ are the integers remaining after $i_1 < \ldots < i_k$ have been deleted from the first n natural numbers, and $j'_1 < \cdots < j'_{n-k}$ are the integers remaining after $j_1 < \cdots < j_k$ have been deleted from the first n natural numbers, $j_{s_1} < \cdots < j_{s_{k-m}}$ are the integers of the set $\{j_1, \ldots, j_k\}$ which are not in the set $\{i_1, \ldots, i_k\}$ and $h_1 < \cdots < h_m$ are the integers in both, $i'_{r'_1} < \cdots < i'_{r'_{k-m}}$ are the integers of the set $\{i'_1, \ldots, i'_{n-k}\}$ which are not in the set $\{j'_1, \ldots, j'_{n-k}\}$ and $h'_1 < \cdots < h'_{n+m-2k}$ are the integers in both.

According to (5.8.3), the set $\{j_{s_1}, \ldots, j_{s_{k-m}}\}$ consists of those integers of $\{j_1, \ldots, j_k\}$ which are not in $\{i_1, \ldots, i_k\}$. In other words, $\{j_{s_1}, \ldots, j_{s_{k-m}}\}$ is the subset of $\{j_1, \ldots, j_k\}$ which is contained in the complement of $\{i_1, \ldots, i_k\}$. It follows, therefore, that

$$\{j_{s_1}, \ldots, j_{s_{k-m}}\} = \{j_1, \ldots, j_k\} \cap \{i'_1, \ldots, i'_{n-k}\}.$$

Similarly, the set $\{i'_{r'_1}, \ldots, i'_{r'_{k-m}}\}$ consists of those integers of $\{i'_1, \ldots, i'_{n-k}\}$ which are not in the set $\{j'_1, \ldots, j'_{n-k}\}$. Hence $\{i'_{r'_1}, \ldots, i'_{r'_{k-m}}\}$ is the subset of $\{i'_1, \ldots, i'_{n-k}\}$ which is contained in the set $\{j_1, \ldots, j_k\}$; that is,

$$\{i'_{r'_1}, \ldots, i'_{r'_{k-m}}\} = \{j_1, \ldots, j_k\} \cap \{i'_1, \ldots, i'_{n-k}\}.$$

Consequently, we have

$$\{i'_{r'_1}, \ldots, i'_{r'_{k-m}}\} = \{j_{s_1}, \ldots, j_{s_{k-m}}\} .$$

Thus, since, by (5.8.3), $j_0 = i_{r'_0}$, $i'_{r'_1} < \ldots < i'_{r'_{k-m}}$, and $j_{s_1} < \ldots < j_{s_{k-m}}$, we have the following corollary.

(5.8.4) <u>Corollary.</u> $j_{s_v} = i'_{r'_v}$ for $v = 0, 1, \ldots, k-m.$

By definition, the sets $\{i_1, \ldots, i_k\}$ and $\{i'_1, \ldots, i'_{n-k}\}$ are complementary subsets of the set consisting of the first n natural numbers and $i'_0 = i_r$. Moreover, by (5.8.4), $j_{s_v} = i'_{r'_v}$. It follows, therefore, that, if we exchange $i_r = i'_0$ of the first subset for $i'_{r'_v} = j_{s_v}$ of the second, then the resulting subsets $\{j_{s_v} i_1 \cdots i_{r-1} i_{r+1} \cdots i_k\}$ and $\{i'_0 i'_1 \cdots i'_{r'_v-1} i'_{r'_v+1} \cdots i'_{n-k}\}$ are likewise complementary subsets of the set consisting of the first n natural numbers. Similarly, it follows from the facts that $j_0 = i_r$, $j_{s_v} = i'_{r'_v}$ and that $\{j_1, \ldots, j_k\}$ and $\{j'_1, \ldots, j'_{n-k}\}$ are complementary subsets of $\{1, \ldots, n\}$, that $\{j_0 j_1 \cdots j_{s_v-1} j_{s_v+1} \cdots j_k\}$ and $\{i'_{r'_v} j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}\}$ are complementary subsets of $\{1, \ldots, n\}$. Thus we have the next corollary.

(5.8.5) <u>Corollary.</u> For each value of v, the sets $\{j_{s_v} i_1 \cdots i_{r-1} i_{r+1} \cdots i_k\}$ and $\{i'_0 i'_1 \cdots i'_{r'_v-1} i'_{r'_v+1} \cdots i'_{n-k}\}$ are complementary subsets of the set consisting of the first n natural numbers; so are the sets $\{j_0 j_1 \cdots j_{s-1} j_{s+1} \cdots j_k\}$ and $\{i'_{r'_v} j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}\}$.

In view of (1.6.10), the next corollary follows immediately from (5.8.5).

(5.8.6) <u>Corollary.</u> For each value of v, if the k-tuples consisting of the integers of the sets $\{j_{s_v} i_1 \cdots i_{r-1} i_{r+1} \cdots i_k\}$ and $\{j_0 j_1 \cdots j_{s_v-1} j_{s_v+1} \cdots j_k\}$ be respectively the $(i_r)_{s_v}$-th and the $(j_{s_v})_0$-th terms of the sequence of k-tuples of the first n natural numbers, arranged in lexicographical order, and if the (n-k)-tuples consisting of the integers of the sets $\{i'_0 \cdots i'_{r'_v-1} i'_{r'_v+1} \cdots i'_{n-k}\}$ and $\{i'_{r'_v} j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}\}$ be respectively the $(i'_{r'_v})_0$-th and the $(j'_{s'})_{r'_v}$-th terms of the sequence of (n-k)-tuples of the first n natural numbers, then, for $N = C(n,k)$,

$$(i_r)_{s_v} + (i'_{r'_v})_0 = (j_{s_v})_0 + (j'_{s'})_{r'_v} = N + 1 \quad .$$

Since $1 \leq i'_1 < \cdots < i'_{n-k} \leq n$, there are r'_v-1 integers of the set $\{i'_1, \ldots, i'_{n-k}\}$, $i'_{r'_v}-1$ integers of

$\{1, \ldots, n\}$, and consequently, since $1 \leq i_1 < \ldots < i_k \leq n$ and $\{i'_1, \ldots, i'_{n-k}\}$ is the complement of $\{i_1, \ldots, i_k\}$ in $\{1, \ldots, n\}$, $i'_{r'_v} - r'_v$ integers of $\{i_1, \ldots, i_k\}$ which are less than $i'_{r'_v}$. But, by (5.8.4), $j_{s_v} = i'_{r'_v}$; hence

$$i_{j_{s_v} - r'_v} < j_{s_v} < i_{j_{s_v} - r'_v + 1} . \tag{5.8.7}$$

Similarly, since, by definition, $j_0 = j'_{s'}$, there are $s'-1$ integers of $\{j'_1, \ldots, j'_{n-k}\}$, j_0-1 integers of $\{1, \ldots, n\}$, and consequently, j_0-s' integers of $\{j_1, \ldots, j_k\}$ which are less than j_0; hence

$$j_{j_0-s'} < j_0 < j_{j_0-s'+1} . \tag{5.8.8}$$

Likewise, since, by definition, $i'_0 = i_r$, there are $r-1$ integers of $\{i_1, \ldots, i_k\}$, i'_0-1 integers of $\{1, \ldots, n\}$, and therefore, i'_0-r integers of $\{i'_1, \ldots, i'_{n-k}\}$ which are less than i'_0; hence

$$i'_{i'_0-r} < i'_0 < i'_{i'_0-r+1} . \tag{5.8.9}$$

Finally, since, by (5.8.4), $j_{s_v} = i'_{r'_v}$, there are s_v-1 integers of $\{j_1, \ldots, j_k\}$, $i'_{r'_v}-1$ integers of $\{1, \ldots, n\}$, and therefore, $i'_{r'_v} - s_v$ integers of $\{j'_1, \ldots, j'_{n-k}\}$ which are less than the integer $i'_{r'_v}$; hence

(5.8.10) $$j'_{i'_{r'_v}} - s_v < i'_{r'_v} < j'_{i'_{r'_v}} - s_v + 1$$

Now since $\{i_1, \ldots, i_k\}$ and $\{i'_1, \ldots, i'_{n-k}\}$ are clearly disjoint sets, there are, for a particular value of v, in view of (5.8.3) and (5.8.4), two possibilities:

either $$(j_{s_v} = i'_{r'_v}) < (j_0 = i'_0 = j'_{s'} = i_r)$$

or $$(j_0 = i'_0 = j'_{s'} = i_r) < (j_{s_v} = i'_{r'_v}) \quad .$$

Therefore, since the X's and Y's are skew-symmetric in their suffixes, it follows from (5.8.7), (5.8.8), (5.8.9), and (5.8.10) respectively that, in the first case,

$X_{j_{s_v} i_1 \cdots i_{r-1} i_{r+1} \cdots i_k}$ is equal to $(-1)^{j_{s_v} - r'_v}$ times

$$X_{i_1 \cdots i_{j_{s_v} - r'_v} \, j_{s_v} \, i_{j_{s_v} - r'_v + 1} \cdots i_{r-1} i_{r+1} \cdots i_k} \, ,$$

$X_{j_0 \cdots j_{s_v - 1} j_{s_v + 1} \cdots j_k}$ is equal to $(-1)^{j_0 - s' - 1}$ times

$$X_{j_1 \cdots j_{s_v - 1} j_{s_v + 1} \cdots j_{j_0 - s'} \, j_0 \, j_{j_0 - s' + 1} \cdots j_k} \, ,$$

$Y_{i'_0 \cdots i'_{r'_v - 1} i'_{r'_v + 1} \cdots i'_{n-k}}$ is equal to $(-1)^{i'_0 - r - 1}$ times

$$Y_{i'_1 \cdots i'_{r'_v - 1} i'_{r'_v + 1} \cdots i'_{i'_0 - r} \, i'_0 \, i'_{i'_0 - r + 1} \cdots i'_{n-k}} \, ,$$

$Y_{i'_{r'_v} j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}}$ is equal to $(-1)^{i'_{r'_v} - s_v}$ times

$$Y_{j'_1 \cdots j'_{i'_{r'_v}-s_v} i'_{r'_v} j'_{i'_{r'_v}-s_v+1} \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}} ,$$

and, in the second case,

$X_{j_{s_v} i_1 \cdots i_{r-1} i_{r+1} \cdots i_k}$ is equal to $(-1)^{j_{s_v} - r'_v - 1}$ times

$$X_{i_1 \cdots i_{r-1} i_{r+1} \cdots i_{j_{s_v} - r'_v} j_{s_v} i_{j_{s_v} - r'_v + 1} \cdots i_k} ,$$

$X_{j_0 \cdots j_{s_v-1} j_{s_v+1} \cdots j_k}$ is equal to $(-1)^{j_0 - s'}$ times

$$X_{j_1 \cdots j_{j_0 - s'} j_0 j_{j_0 - s'+1} \cdots j_{s_v-1} j_{s_v+1} \cdots j_k} ,$$

$Y_{i'_0 \cdots i'_{r'_v-1} i'_{r'_v+1} \cdots i'_{n-k}}$ is equal to $(-1)^{i'_0 - r}$ times

$$Y_{i'_1 \cdots i'_{i'_0-r} i'_0 i'_{i'_0-r+1} \cdots i'_{r'_v-1} i'_{r'_v+1} \cdots i'_{n-k}} ,$$

$Y_{i'_{r'_v} j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{n-k}}$ is equal to $(-1)^{i'_{r'_v} - s_v - 1}$ times

$$Y_{j'_1 \cdots j'_{s'-1} j'_{s'+1} \cdots j'_{i'_{r'_v} - s_v} i'_{r'_v} j'_{i'_{r'_v} + s_v+1} \cdots j'_{n-k}} .$$

Hence it follows from (5.8.3) that, under the correspondence described in 5.7 and in the notation of (5.8.6), we have

$$\sum_{\nu=0}^{k-m} (-1)^{r+s_v+j_{s_v}-r'_v+j_0-s'-1} X_{(i_r)_{s_v}} X_{(j_{s_v})_0} = 0 \quad \text{and}$$

$$\sum_{\nu=0}^{k-m} (-1)^{r'_v+s'+i'_0-r+i'_{r'_v}-s_v-1} Y_{(i'_{r'_v})_0} Y_{(j'_{s'})_{r'_v}} = 0 .$$

Since, by (5.8.3) and (5.8.4), $i'_0 = j_0$ and $i'_{r'_v} = j_{s_v}$, these relations can be put in the respectively equivalent forms

$$\sum_{\nu=0}^{k-m} (-1)^{r+s'+r'_v+s_v+j_0+j_{s_v}+1} X_{(i_r)_{s_v}} X_{(j_{s_v})_0} = 0$$

and

$$\sum_{\nu=0}^{k-m} (-1)^{r+s'+r'_v+s_v+j_0+j_{s_v}+1} Y_{(i'_{r'_v})_0} Y_{(j'_{s'})_{r'_v}} = 0 \quad .$$

Comparing these with (5.7.1), we see that, in view of (5.8.6), the next corollary is true.

(5.8.11) <u>Corollary.</u> If $Q^{n,k}_{rij}$ and $Q^{n,n-k}_{s'i'j'}$ are dual special Grassmann matrices, then it is true for their entries that

$$\left[Q^{n,k}_{rij}\right]^{(j_{s_v})_0}_{(i_r)_{s_v}} = \left[Q^{n,k}_{rij}\right]^{(i_r)_{s_v}}_{(j_{s_v})_0} = (-1)^{r+s'+r'_v+s_v+j_0+j_{s_v}+1}$$

$$= \left[Q^{n,n-k}_{s'i'j'}\right]^{N+1-(i_r)_{s_v}}_{N+1-(j_{s_v})_0} = \left[Q^{n,n-k}_{s'i'j'}\right]^{N+1-(j_{s_v})_0}_{N+1-(i_r)_{s_v}}$$

for $v = 0,1,\ldots,k-m$, and

$$\left[Q^{n,k}_{rij}\right]^{\beta}_{\alpha} = \left[Q^{n,k}_{rij}\right]^{\alpha}_{\beta} = 0 = \left[Q^{n,n-k}_{s'i'j'}\right]^{N+1-\alpha}_{N+1-\beta} = \left[Q^{n,n-k}_{s'i'j'}\right]^{N+1-\beta}_{N+1-\alpha}$$

otherwise, where $N = C(n,k)$, $j'_{s'} = i_r = j_0$, $i'_{r'_v} = j_{s_v}$.

The next theorem is a direct consequence of (5.8.11).

(5.8.12) <u>Theorem.</u> If $Q^{n,k}_{rij}$ and $Q^{n,n-k}_{s'i'j'}$ are dual special Grassmann matrices then

$$\left[Q^{n,k}_{rij}\right]^{\beta}_{\alpha} = \left[Q^{n,k}_{rij}\right]^{\alpha}_{\beta} = \left[Q^{n,n-k}_{s'i'j'}\right]^{N+1-\alpha}_{N+1-\beta} = \left[Q^{n,n-k}_{s'i'j'}\right]^{N+1-\beta}_{N+1-\alpha},$$

where $\alpha, \beta = 1, \ldots, (N = C(n,k))$.

Now, if we let $H_{n,k}$ denote the NxN matrix whose entries are zero except those along the secondary diagonal, each of these being unity, it is easy to prove that the following theorem is true.

(5.8.13) <u>Theorem.</u> If $H_{n,k}$ be the NxN matrix with all elements zero except those in the secondary diagonal, these elements being unity, then the transform of each special Grassmann matrix $Q^{n,k}_{rij}$ by $H_{n,k}$ is the dual special Grassmann matrix $Q^{n,n-k}_{s'i'j'}$, that is,

$$H^{-1}_{n,k}\, Q^{n,k}_{rij}\, H_{n,k} = Q^{n,n-k}_{s'i'j'} \quad .$$

To prove (5.8.13), we note first that $H^{-1}_{n,k} = H_{n,k}$. Thus, we have, using the Kronecker delta,

$$H^{-1}_{n,k}\ Q^{n,k}_{rij}\ H_{n,k} = \left[\delta^{N+1-\alpha}_{\beta}\right]\left[\left[Q^{n,k}_{rij}\right]^{\alpha}_{\beta}\right]\left[\delta^{\alpha}_{N+1-\beta}\right]$$

$$= \left[\left[Q^{n,k}_{rij}\right]^{N+1-\alpha}_{N+1-\beta}\right]$$

$$= Q^{n,n-k}_{s'i'j'} \qquad \text{by (5.8.12).}$$

Now, let us consider the transform of an arbitrary special Grassmann matrix $Q^{n,k}_{rij}$ by the matrix $G_{n,k}$ defined by (1.7.5). We have, by (1.7.5), (1.8.2) and (1.8.3), for the α,β-th element of $G^{-1}_{n,k}\ Q^{n,k}_{rij}\ G_{n,k}$,

$$\left[G^{-1}_{n,k} Q^{n,k}_{rij} G_{n,k}\right]^{\alpha}_{\beta} = \sum_{\mu,\nu=1}^{N} (-)^{p_\alpha + p_\beta}\, \delta^{N+1-\alpha}_{\mu} \left[Q^{n,k}_{rij}\right]^{\mu}_{\nu} \delta^{\nu}_{N+1-\beta}$$

$$= (-)^{p_\alpha + p_\beta} \left[Q^{n,k}_{rij}\right]^{N+1-\alpha}_{N+1-\beta}$$

$$= (-)^{p_\alpha + p_\beta} \left[Q^{n,n-k}_{s'i'j'}\right]^{\alpha}_{\beta}, \qquad \text{by (5.8.12),}$$

where $p_\alpha = \alpha_1 + \dots + \alpha_k$ and $p_\beta = \beta_1 + \dots + \beta_k$ when $(\alpha_1, \dots, \alpha_k)$ and $(\beta_1, \dots, \beta_k)$ are respectively the α-th and β-th k-tuples of the sequence of k-tuples of the first n natural numbers taken in lexicographical order. Now we know, by (5.8.11), that the only non-zero elements of $Q^{n,k}_{rij}$ are those for which

$$\alpha, \beta = (i_r)_{s_v}, (j_{s_v})_0 , \quad v = 0,1,\ldots,k-m ,$$

and their symmetric images

$$\beta, \alpha = (j_{s_v})_0 , (i_r)_{s_v} , \quad v = 0,1,\ldots,k-m .$$

For these, we have, by (5.8.6), for each value of v,

$$p_{(i_r)_{s_v}} + p_{(j_{s_v})_0} = (j_{s_v} + i_1 + \cdots + i_{r-1} + i_{r+1} + \cdots + i_k)$$
$$+ (j_0 + \cdots + j_{s_v-1} + j_{s_v+1} + \cdots + j_k).$$

But, by definition, $j_0 = i_r$, hence

$$p_{(i_r)_{s_v}} + p_{(j_{s_v})_0} = (i_1 + \cdots + i_k) + (j_1 + \cdots + j_k)$$
$$= p_i + p_j .$$

Obviously, it is also true that

$$p_{(j_{s_v})_0} + p_{(i_r)_{s_v}} = p_j + p_i .$$

We see, therefore, that the sign for all non-vanishing elements of the transformed matrix is the same, namely,

$$(-)^{p_i + p_j} .$$

Since it is immaterial what the sign is for the zero elements of the transformed matrix, we may replace $(-)^{p_\alpha + p_\beta}$ by $(-)^{p_i + p_j}$ in these entries. We will have then

$$\left[G_{n,k}^{-1} \; Q_{rij}^{n,k} \; G_{n,k}\right]_\beta^\alpha = (-)^{p_i + p_j} \left[Q_{s'i'j'}^{n,n-k}\right]_\beta^\alpha$$

for all α, β. Hence we have the next theorem.

(5.8.14) Theorem. Let $(i_1,\dots,i_k)$ and $(j_1,\dots,j_k)$ be the i-th and j-th k-tuples, respectively, of the sequence of k-tuples of the first n natural numbers taken in lexicographical order; let $p_i = i_1+\dots+i_k$ and $p_j = j_1+\dots+j_k$; and let $G_{n,k}$ be the matrix defined by (1.7.5). Then the transform of an arbitrary special Grassmann matrix $Q^{n,k}_{rij}$ by $G_{n,k}$ is $(-1)^{p_i+p_j}$ times the dual special Grassmann matrix $Q^{n,n-k}_{s'i'j'}$; that is,

$$G^{-1}_{n,k}\, Q^{n,k}_{rij}\, G_{n,k} = (-)^{p_i + p_j}\, Q^{n,n-k}_{s'i'j'} \quad .$$

Let $Q^{n,k}$ be any matrix of the set $\{Q^{n,k}\}$ of Grassmann matrices for k-vectors. Then, by (5.7.5), there exists a finite number of elements a_{rij} of K such that

$$Q^{n,k} = \sum_{rij} a_{rij}\, Q^{n,k}_{rij} \quad ,$$

where the summation extends over the set of special Grassmann matrices. Then, by (5.8.13),

$$\begin{aligned} H^{-1}_{n,k}\, Q^{n,k}\, H_{n,k} &= H^{-1}_{n,k} \left[\sum_{rij}' a_{rij}\, Q^{n,k}_{rij} \right] H_{n,k} \\ &= \sum_{rij} a_{rij}\, H^{-1}_{n,k}\, Q^{n,k}_{rij}\, H_{n,k} \\ &= \sum_{rij} a_{rij}\, Q^{n,n-k}_{s'i'j'} \\ &= \sum_{s'i'j'} b_{s'i'j'}\, Q^{n,n-k}_{s'i'j'} \quad ; \end{aligned}$$

and hence, by (5.7.5), there exists some matrix, say $Q^{n,n-k}$, of the set $\{Q^{n,n-k}\}$ such that

$$H^{-1}_{n,k}\ Q^{n,k}\ H_{n,k} = Q^{n,n-k} .$$

Since the argument is reversible, we have the next theorem.

(5.8.15) Theorem. The NxN matrix $H_{n,k}$ with all elements zero except those in the secondary diagonal, these elements being unity, transforms the set of Grassmann matrices for k-vectors into the set of Grassmann matrices for (n-k)-vectors; that is,

$$H^{-1}_{n,k}\ \{Q^{n,k}\}\ H_{n,k} = \{Q^{n,n-k}\} .$$

Similarly, it follows from (5.8.14) that the next theorem is true.

(5.8.16) Theorem. The NxN matrix $G_{n,k}$ defined by (1.7.5) transforms the set of Grassmann matrices for k-vectors into the set for (n-k)-vectors; that is,

$$G^{-1}_{n,k}\ \{Q^{n,k}\}\ G_{n,k} = \{Q^{n,n-k}\} .$$

Now, since $H^{-1}_{n,k} = H'_{n,k}$ and, by (1.8.2), $G^{-1}_{n,k} = G'_{n,k}$, we have the following corollaries.

(5.8.17) Corollary. $H'_{n,k}\ \{Q^{n,k}\}\ H_{n,k} = \{Q^{n,n-k}\}$.

(5.8.18) Corollary. $G'_{n,k}\ \{Q^{n,k}\}\ G_{n,k} = \{Q^{n,n-k}\}$.

<u>5.9. Example (n = 4, k = 2, n-k = 2).</u> In accordance with the discussion of section 5.8, the determining Grassmann quadratic relation for the special Grassmann matrix

$$Q_1^{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is

$$(5.9.1) \qquad X_{12}\, X_{34} = X_{13}\, X_{24} + X_{14}\, X_{32}$$

and the dual quadratic relation is

$$(5.9.2) \qquad Y_{34}\, Y_{12} = Y_{24}\, Y_{13} + Y_{32}\, Y_{14},$$

which determines the same special Grassmann matrix. This shows that $Q_1^{4,2}$ is self-dual.

The set of special Grassmann matrices in this case contains only three distinct matrices, namely,

$$Q_1^{4,2}\ , \quad -Q_1^{4,2}\ , \quad \text{and} \quad 0_6\ ,$$

where 0_6 is the zero matrix of order six.

Moreover, by definition, we have

$$H_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad G_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Now it is very easy to verify (5.8.13) and (5.8.14) in this case.

<u>5.10 Example ($n = 5$, $k = 2$, $n-k = 3$).</u> In this case, one finds five distinct Grassmann quadratic relations for 2-vectors and five dual relations for 3-vectors. They are:

$$(5.10.1) \qquad \begin{aligned} X_{12} X_{34} - X_{13} X_{24} + X_{14} X_{23} &= 0 \quad , \\ X_{12} X_{35} - X_{13} X_{25} + X_{15} X_{23} &= 0 \quad , \\ X_{12} X_{45} - X_{14} X_{25} + X_{15} X_{24} &= 0 \quad , \\ X_{13} X_{45} - X_{14} X_{35} + X_{15} X_{34} &= 0 \quad , \\ X_{23} X_{45} - X_{24} X_{35} + X_{25} X_{34} &= 0 \quad , \end{aligned}$$

and

$$(5.10.2) \qquad \begin{aligned} Y_{345} Y_{125} - Y_{245} Y_{135} + Y_{235} Y_{145} &= 0 \quad , \\ Y_{345} Y_{124} - Y_{245} Y_{134} + Y_{234} Y_{145} &= 0 \quad , \\ Y_{345} Y_{123} - Y_{235} Y_{134} + Y_{234} Y_{135} &= 0 \quad , \\ Y_{245} Y_{123} - Y_{235} Y_{124} + Y_{234} Y_{125} &= 0 \quad , \\ Y_{145} Y_{123} - Y_{135} Y_{124} + Y_{134} Y_{125} &= 0 \quad . \end{aligned}$$

Under the correspondence described in Section 5.7, these become respectively

$$
(5.10.3)\qquad
\begin{aligned}
X_1 X_8 - X_2 X_6 + X_3 X_5 &= 0 \quad ,\\
X_1 X_9 - X_2 X_7 + X_4 X_5 &= 0 \quad ,\\
X_1 X_{10} - X_3 X_7 + X_4 X_6 &= 0 \quad ,\\
X_2 X_{10} - X_3 X_9 + X_4 X_8 &= 0 \quad ,\\
X_5 X_{10} - X_6 X_9 + X_7 X_8 &= 0 \quad ,
\end{aligned}
$$

and .....

$$
(5.10.4)\qquad
\begin{aligned}
Y_{10} Y_3 - Y_9 Y_5 + Y_8 Y_6 &= 0 \quad ,\\
Y_{10} Y_2 - Y_9 Y_4 + Y_7 Y_6 &= 0 \quad ,\\
Y_{10} Y_1 - Y_8 Y_4 + Y_7 Y_5 &= 0 \quad ,\\
Y_9 Y_1 - Y_8 Y_2 + Y_7 Y_3 &= 0 \quad ,\\
Y_6 Y_1 - Y_5 Y_2 + Y_4 Y_3 &= 0 \quad .
\end{aligned}
$$

From these quadratic forms, we obtain the following special Grassmann matrices:

$$
Q_1^{5,2} = \begin{bmatrix}
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}, \quad
Q_1^{5,3} = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0
\end{bmatrix},
$$

$$
Q_2^{5,2} = \begin{bmatrix}
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}, \quad
Q_2^{5,3} = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0
\end{bmatrix},
$$

$$Q_3^{5,2} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0 \end{bmatrix}, \quad Q_3^{5,3} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$Q_4^{5,2} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0 \end{bmatrix}, \quad Q_4^{5,3} = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$Q_5^{5,2} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0 \end{bmatrix}, \quad Q_5^{5,3} = \begin{bmatrix} 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

Furthermore, we have, by (1.7.5),

$$G_{5,2} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0 \end{bmatrix}, \quad G_{5,3} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

and, by definition,

$$H_{5,2} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Comparing (5.10.1) with (5.10.2) and (5.10.3) with (5.10.4), we see that one obtains the quadratic relations for 3-vectors in reversed lexicographical order when he dualizes the relations, taken in lexicographical order, for 2-vectors.

The complete set of special Grassmann matrices for 2-vectors is seen to consist of the eleven distinct matrices $\pm\, Q_i^{5,2}$ $(i = 0, 1, 2, 3, 4, 5)$, where $Q_0^{5,2}$ is to be taken as the zero matrix of order ten. Similarly, it is clear that the complete set of special Grassmann matrices for 3-vectors consists of the eleven distinct matrices $\pm\, Q_i^{5,3}$ $(i = 0, 1, 2, 3, 4, 5)$, where $Q_0^{5,3}$ is to be taken as the zero matrix of order ten also. Furthermore, it is evident that, for each value of i, $-\, Q_i^{5,2}$ and $-\, Q_i^{5,3}$, as well as $Q_i^{5,2}$ and $Q_i^{5,3}$, are duals.

Again, it is a simple matter to verify (5.8.13) and (5.8.14).

Chapter 6

Linear Transformations

By (3.5.11), we know that the components of every k-cell are left unchanged by any translation. In this chapter we shall consider (1) those linear transformations on the base space $I_n(K)$ which leave the components of k-cells unchanged and (2) those which leave the norms or "volumes" of k-cells unchanged. Throughout this chapter we shall assume $0 < k < n$ for simplicity.

<u>6.1. Components of Transformed k-Cells.</u> Let $A = \left[a^i_j\right]$ be the $n \times n$ matrix of an arbitrary linear transformation on $I_n(K)$, and let X be the matrix of edges at the initial vertex of any k-cell $((x^0,\ldots,x^k))$ in $I_n(K)$. The transformation A carries the k-cell $((x^0,\ldots,x^k))$ into the k-cell $((x^0A,\ldots,x^kA))$, where

$$x^iA_j = \sum_{\kappa=1}^{n} x^i_k\, a^k_j \quad , \qquad\qquad j = 1, \ldots, n.$$

And, in view of (3.1.4), the matrix of edges at the initial vertex of $((x^0A,\ldots,x^kA))$ is therefore

$$\left[x^iA_j - x^0A_j\right] = \left[\sum_{\kappa=1}^{n} x^i_k\, a^k_j - \sum_{\kappa=1}^{n} x^0_k\, a^k_j\right]$$

$$= \left[\sum_{\kappa=1}^{n} (x^i_k - x^0_k)\, a^k_j\right]$$

$$= X\,A \quad .$$

Hence we have the following theorem.

(6.1.1) Theorem. If A is the matrix of a linear transformation on $L_n(K)$, then, for every k-cell $((x^0,\ldots,x^k))$ in $L_n(K)$, the matrix of edges at the initial vertex of $((x^0A,\ldots,x^kA))$ is the $k\times n$ matrix XA, where X is the matrix of edges at the initial vertex of $((x^0,\ldots,x^k))$.

Using (3.3.5), we obtain the following theorem as a corollary to (6.1.1).

(6.1.2) Theorem. The components of $((x^0A,\ldots,x^kA))$ are the elements of the $1\times C(n,k)$ matrix $(1/k!)X^{(k)}A^{(k)}$; in other words, the k-vectors determined by the k-cells in $L_n(K)$ are transformed by the k-th compound of the matrix of any linear transformation on $L_n(K)$.

Considerations of rank then lead us to the next theorem.

(6.1.3) Theorem. A linear transformation on $L_n(K)$ carries proper k-cells into proper k-cells if, and only if, it is a non-singular linear transformation.

Now let A be the matrix of any linear transformation on the associated N-space which transforms every k-vector in $L_N(K)$ into the null vector. Then, in particular, A transforms the basis vectors (5.3.1) into the null vector since, by (5.3.7), they are k-vectors. Hence we must have

$$[0, \dots, 0] = E^i A$$
$$= [0, \dots, 1, \dots, 0] A$$
$$= [A^i_1, \dots, A^i_N] \quad ,$$

for $i = 1, \dots, N$. It follows, therefore, that

$$A^i_1 = A^i_2 = \dots = A^i_N = 0 \quad ,$$

for $i = 1, \dots, N$; and hence A is the $N \times N$ zero matrix. We have proved the next theorem, since the converse is obviously true.

(6.1.4) Theorem. A linear transformation on the associated N-space carries all k-vectors into the null vector if, and only if, its matrix is the $N \times N$ zero matrix.

Now let A be the matrix of any linear transformation on $L_n(K)$ which transforms every k-cell with initial vertex at the origin into a null k-cell. Then, since every k-vector in the associated N-space is, by (5.4.1) and (5.4.3), the image of some k-cell with initial vertex at the origin in $L_n(K)$, it follows from (6.1.2) that the k-th compound $A^{(k)}$ of A carries every k-vector into the null vector, and consequently, from (6.1.4) that $A^{(k)}$ is the $N \times N$ zero matrix, which means that A has rank less than k. Conversely, if A has rank less than k, it is clear, by (6.1.2), that A carries all k-cells into null k-cells. Hence we have the following theorem.

(6.1.5) <u>Theorem.</u> A linear transformation on $L_n(K)$ transforms all k-cells with initial vertex at the origin into null k-cells if, and only if, its matrix has rank less than k.

Since a transformation which transforms all k-cells into null k-cells must, perforce, transform all k-cells with initial vertex at the origin into null k-cells, it follows from (6.1.5) as a corollary that such a linear transformation has a matrix of rank less than k. Hence we have the corollary:

(6.1.6) <u>Corollary.</u> A linear transformation on $L_n(K)$ transforms all k-cells into null k-cells if, and only if, its matrix has rank less than k.

Let λ_o be an arbitrary non-zero element of K and let K be algebraically closed; and consider the identity

$$X^{(k)}A^{(k)} = \lambda_o X^{(k)} \quad ,$$

which can be written in the form

$$X^{(k)} \left[A^{(k)} - \lambda_o I^{(k)}\right] = 0 \quad .$$

It follows from (6.1.6) that

$$A^{(k)} = \lambda_o I^{(k)} = \left[\lambda_o^{1/k}\, I\right]^{(k)} \quad ;$$

and consequently, in view of (1.4.11),

$$A = \omega \lambda_0^{1/k} I \qquad (\omega \text{ is a k-th root of unity}),$$

since, according to (6.1.3), A is non-singular. This, along with (6.1.2), proves the next theorem.

(6.1.7) Theorem. Whenever K is algebraically closed, those, and only those, linear transformations whose matrices are of the form $A = \omega \lambda I$, where ω is any k-th root of unity and λ is an element of K, carry every k-cell in $L_n(K)$ into another k-cell whose components are proportional to those of its ancestor.

Taking $\lambda = 1$ in (6.1.7) and observing that the set of $n \times n$ matrices of the form ωI is a cyclic group under matrix multiplication, we have our next theorem.

(6.1.8) Theorem. The set of linear transformations under which the components of all k-cells in $L_n(K)$ are invariant form a cyclic group of order k, with respect to transformation multiplication. In particular, the only real transformations under which the components of k-cells are invariant are the identity transformation, when k is odd, and the identity and its negative, when k is even.

6.2. Inner Product of Transformed k-Cells (and k-Vectors). Let $((x^0, \ldots, x^k))$, $((y^0, \ldots, y^k))$ be arbitrary k-cells and A the matrix of any linear transformation in C_n (or R_n). Then we have, by (4.4.3),

$$\begin{bmatrix} x^0A & y^0 \\ \vdots & \vdots \\ x^kA & y^k \end{bmatrix} = (1/k!)^2 \left| [XA]Y^* \right| = (1/k!)^2 \left| X[YA^*]^* \right| = \begin{bmatrix} x^0 & y^0A^* \\ \vdots & \vdots \\ x^k & y^kA^* \end{bmatrix}$$

and, similarly,

$$\begin{bmatrix} x^0 & y^0A \\ \vdots & \vdots \\ x^k & y^kA \end{bmatrix} = \begin{bmatrix} x^0A^* & y^0 \\ \vdots & \vdots \\ x^kA^* & y^k \end{bmatrix} .$$

This proves the following theorem.

(6.2.1) <u>Theorem.</u> For every pair of k-cells and every linear transformation over C_n(or R_n), it is true that

$$\begin{bmatrix} x^0A & y^0 \\ \vdots & \vdots \\ x^kA & y^k \end{bmatrix} = \begin{bmatrix} x^0 & y^0A^* \\ \vdots & \vdots \\ x^k & y^kA^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x^0 & y^0A \\ \vdots & \vdots \\ x^k & y^kA \end{bmatrix} = \begin{bmatrix} x^0A^* & y^0 \\ \vdots & \vdots \\ x^kA^* & y^k \end{bmatrix} .$$

Suppose that for every pair of k-vectors X, Y in the associated N-space of C_n(or R_n) it is true that

$$(XA, Y) = 0$$

for a particular linear transformation with matrix A. Then, in particular, for the basis vectors (5.3.1), which are k-vectors by (5.3.7), it is true that

$$(E^iA, E^j) = 0.$$

Therefore, by (4.1.1), it is true that

$$a^i_j = 0, \qquad i,j = 1, \ldots, N,$$

where a_j^i is the i,j-th element of A. Hence A is the zero matrix. Conversely, if A is the zero matrix, XA is, by (6.1.4), the null vector; and hence, it is easy to see by (4.1.1) that $(XA, Y) = 0$ for every pair X, Y. This proves the next theorem.

(6.2.2) Theorem. Let A be the matrix of a linear transformation on the associated N-space of C_n(or R_n). Then a necessary and sufficient condition that the inner product of XA with Y vanish for every pair X, Y of k-vectors in C_N(or R_N) is that A be the zero matrix.

Suppose that for every pair $((\theta, x^1, \ldots, x^k))$, $((\theta, y^1, \ldots, y^k))$ of k-cells with initial vertex at the origin in C_n(or R_n) it is true that

$$\left[\begin{array}{c|c} \theta A & \theta \\ x^1 A & y^1 \\ \vdots & \vdots \\ x^k A & y^k \end{array}\right] = 0$$

for a particular linear transformation with matrix A. Replacing $((\theta, \ldots, y^k))$ by $((\theta A, \ldots, x^k A))$, we have by (4.1.16),

$$\left\| \begin{array}{c} \theta A \\ x^1 A \\ \vdots \\ x^k A \end{array} \right\|^2 = 0$$

identically. It follows from (4.3.10) that A carries every k-cell with initial vertex at the origin into a null k-cell; and hence, by (6.1.5), that A has rank less than k. On the other hand, if A has rank less than k, $((\theta A,\ldots,x^k A))$ is, by (6.1.5), a null k-cell for every k-cell $((\theta,x^1,\ldots,x^k))$ with initial vertex at the origin. Hence, by (4.1.15),

$$\left[\begin{array}{c|c} \theta A & \theta \\ x^1 A & y^1 \\ \vdots & \vdots \\ x^k A & y^k \end{array}\right]$$

vanishes identically. This proves the next theorem.

(6.2.3) <u>Theorem.</u> If A is the matrix of a linear transformation on C_n(or R_n), then a necessary and sufficient condition that the generalized inner product of $((\theta A, x^1 A,\ldots,x^k A))$ with $((\theta,y^1,\ldots,y^k))$ vanish for every pair $((\theta,x^1,\ldots,x^k))$, $((\theta,y^1,\ldots,y^k))$ of k-cells with initial vertex at the origin in C_n(or R_n) is that A have rank less than k.

It is easy to see that (6.2.3) leads to the corollary:

(6.2.4) <u>Corollary.</u> The necessary and sufficient condition that

$$\left[\begin{array}{c|c} x^0 A & y^0 \\ \vdots & \vdots \\ x^k A & y^k \end{array}\right]$$

vanish identically is that A have rank less than k.

We pause here to make the following remarks. In view of (6.1.2), it follows from (6.1.4) that the following is true.

(6.2.5) <u>Remark:</u> Let A be an $N\times N$ matrix, then $X^{(k)}A$ is a zero matrix for every $k\times n$ matrix X if, and only if, A is the zero matrix.

Now it follows from (6.2.5) that the next remark is true.

(6.2.6) <u>Remark:</u> The matrix equation

$$X^{(k)}\ A\left[Y^{*}\right]^{(k)} = 0$$

holds identically for every pair X, Y of $k\times n$ matrices if, and only if, A is the $N\times N$ zero matrix.

Furthermore, (6.2.6) implies both (6.2.2) and (6.2.3).

The next theorem requires a proof by "mathematical induction".

(6.2.7) <u>Theorem.</u> Let A be the matrix of a linear transformation on the associated N-space, C_N, of C_n. Then the vanishing of the inner product (XA, X) for every k-vector in C_N implies the vanishing of (XA, Y) for every pair X, Y of k-vectors in C_N.

In order to facilitate the proof, we introduce some notation. Let X denote the k-vector determined by the k-cell $((\theta, x^1, \ldots, x^k))$ and $X\{y^i\}$, $0 \leq i \leq k$, denote the k-vector determined by the k-cell $((\theta, y^1, \ldots, y^i, x^{i+1}, \ldots, x^k))$, where we understand $X\{y^0\}$ and $X\{y^k\}$ respectively to denote the k-vectors determined by $((\theta, x^1, \ldots, x^k))$ and $((\theta, y^1, \ldots, y^k))$. First we note that $(XA, X) \equiv 0$ implies $(XA, X\{y^0\}) \equiv 0$ trivially since $X \equiv X\{y^0\}$ by definition. We perform an induction on the integer i, by showing that if $(XA, X) \equiv 0$ implies, for $0 < i \leq k$,

$$(XA, X\{y^0\}) \equiv (XA, X\{y^1\}) \equiv \ldots \equiv (XA, X\{y^{i-1}\}) \equiv 0$$

then $(XA, X) \equiv 0$ also implies $(XA, X\{y^i\}) \equiv 0$. Let us, therefore, assume the induction hypothesis to be true; and consider the inner product

$$\left[\left(\begin{matrix} \theta \\ x^1 \\ \vdots \\ x^{i-1} \\ ax^i + by^i \\ x^{i+1} \\ \vdots \\ x^k \end{matrix} \right) A \;\middle|\; \left(\begin{matrix} \theta \\ y^1 \\ \vdots \\ y^{i-1} \\ ax^i + by^i \\ x^{i+1} \\ \vdots \\ x^k \end{matrix} \right) \right],$$

where a and b are complex numbers to be determined later. Clearly, all inner products of this form vanish under the induction hypothesis. Hence, if we use (3.4.11), (3.4.12), (4.2.2), and (4.2.3) to expand this inner product, we obtain

$$a\,\bar{a}\,(X\,A,\ X\{y^{i-1}\}) + a\,\bar{b}\,(X\,A,\ X\{y^{i}\})$$
$$+ b\,\bar{a}\,(\underline{X}\,A,\ X\{y^{i-1}\}) + b\,\bar{b}\,(\underline{X}\,A,\ \underline{X}\{y^{i-1}\}) \equiv 0,$$

where $\underline{X}$ denotes the k-vector determined by the k-cell $((\theta,x^1,\ldots,x^{i-1},y^i,x^{i+1},\ldots,x^k))$. Now it follows from the induction hypothesis that the first and last terms of the left member of the last identity vanish identically. Therefore we have

$$a\,\bar{b}\,(X\,A,\ X\{y^{i}\}) + b\,\bar{a}\,(\underline{X}\,A,\ X\{y^{i-1}\}) \equiv 0,$$

where we are still at liberty to assign any values we desire to a and b. If we first set $a = b = 1$ and then set $a = \imath$ $(\imath^2 = -1)$, $b = 1$, we obtain the two identities:

$$(X\,A,\ X\{y^{i}\}) + (\underline{X}\,A,\ X\{y^{i-1}\}) \equiv 0,$$

$$\imath\,(X\,A,\ X\{y^{i}\}) - \imath\,(\underline{X}\,A,\ X\{y^{i-1}\}) \equiv 0.$$

Dividing $\imath$ out of the second identity and then adding the two identities, yields the desired identity, namely,

$$(X\,A,\ X\{y^{i}\}) \equiv 0.$$

This completes the induction, since we have already shown that $(XA,\ X) \equiv 0$ implies $(XA,\ X\{y^0\}) \equiv 0$. Therefore (6.2.7) is true by "mathematical induction", since $(XA,\ X\{y^k\}) \equiv (XA,\ Y)$. It should be noted that the crux of the proof depended upon the assumption that the base space was a <u>complex</u> space.

Applying (6.2.2) to (6.2.7), we deduce the next theorem, in view of (5.3.5).

(6.2.8) <u>Theorem.</u> If A be the matrix of a linear transformation on the associated N-space of C_n, then a necessary and sufficient condition that the inner product of XA with X vanish for every k-vector X in C_N is that A be the $N\times N$ zero matrix.

Now suppose that the generalized inner product of $((\theta A, x^1 A, \ldots, x^k A))$ with $((\theta, x^1, \ldots, x^k))$ vanishes identically for every k-cell $((\theta, x^1, \ldots, x^k))$ with initial vertex at the origin in C_n. Then

$$(X^{(k)}A^{(k)}, X^{(k)}) \equiv 0,$$

where $X^{(k)}$ is the k-th compound of the matrix of edges at the initial vertex of $((\theta, x^1, \ldots, x^k))$; hence, by (4.1.5) with n replaced by N, we have

$$((1/k!)\, X^{(k)}A^{(k)}, (1/k!)\, X^{(k)}) \equiv 0.$$

Therefore it follows from (6.1.2), (5.3.5), and (6.2.8) that $A^{(k)}$ is the $N\times N$ zero matrix; and hence A has rank less than k. Conversely, if A has rank less than k, then, by (6.1.5), $((\theta A, x^1 A, \ldots, x^k A))$ is a null k-cell for all k-cells $((\theta, x^1, \ldots, x^k))$ with initial vertex at the origin in C_n. It is clear, then by (4.1.1), that the inner product vanishes identically for k-cells with initial vertex at the origin. Hence we have the next theorem.

(6.2.9) Theorem. If A be the matrix of a linear transformation on C_n, then a necessary and sufficient condition that the generalized inner product of $((\theta A, x^1 A, \ldots, x^k A))$ vanish identically for every k-cell $((\theta, x^1, \ldots, x^k))$ with initial vertex at the origin in C_n is that A should have rank less than k.

Since every k-cell in C_n is, by (5.4.3), equivalent to some k-cell with initial vertex at the origin, we obtain an immediate corollary to (6.2.9).

(6.2.10) Corollary. When the base space is C_n,

$$\left[\begin{array}{c|c} x^0 A & x^0 \\ \vdots & \vdots \\ x^k A & x^k \end{array}\right]$$

vanishes identically if, and only if, A has rank less than k.

To obtain (6.2.8) we had to make very decided use of the fact that we were working in a complex space. Hence it is to be expected that the same theorem is unobtainable in real space. Indeed, it turns out that (6.2.8) is not true in R_N. However, we can obtain sufficient conditions and certain necessary conditions. To begin with, we consider some special cases. For instance, let us first prove the following theorem.

(6.2.11) <u>Theorem.</u> If A be the matrix of a linear transformation on the associated N-space of R_n, then a necessary and sufficient condition that the inner product of XA with X vanish for every 1-vector (or (n-1)-vector) X in $R_N = R_n$ is that the $n \times n$ matrix A be skew-symmetric.

We note first that, since $C(n,1) = n$ (or $C(n,n-1) = n$), $R_N = R_n$. Moreover, since the Grassmann quadratic relations impose no restriction when $k = 1$ (or $k = n-1$), the set of k-vectors fills out the associated N-space and hence, in view of (5.4.7), we need only prove that

$$(xA,x) = 0$$

if, and only if $A^* = -A$, where, since A is a real matrix, A^* denotes the transposed matrix of A. Firstly, we show the condition to be sufficient. If $A^* = -A$, then $A^* + A = 0_n$, and, using (4.1.1), (4.1.4), and (4.1.14), we have

$$\begin{aligned} 0 &= (x\,0_n,x) \\ &= (x[A^*+ A],x) \\ &= (xA^*+ xA,x) \end{aligned}$$

$$= (xA^*,x) + (xA,x)$$

$$= (x,xA) + (xA,x)$$

$$= (xA,x) + (xA,x)$$

$$= 2(xA,x)$$

for every vector x in R_n, which proves the condition sufficient. Conversely, if

$$(xA,x) = 0$$

for every vector x in R_n, we have

$$0 = ((x+y)A,x+y)$$

$$= (xA+yA,x+y)$$

$$= (xA,x) + (yA,x) + (xA,y) + (yA,y)$$

$$= (y,xA^*) + (xA,y)$$

$$= (xA^*,y) + (xA,y)$$

$$= (xA^*+xA,y)$$

$$= (x[A^*+ A],y)$$

for every pair of vectors in R_n. Hence, by (6.2.2),

$$A^* + A = 0_n;$$

in other words, A is skew-symmetric. This completes the proof of (6.2.11).

Now it is clear from the foregoing proof that the condition $A^* = -A$ is also sufficient that $(XA,X) = 0$ for all k-vectors X even when $1 < k < n-1$. But we can find a weaker condition for these cases. Therefore, it is obvious that the condition $A^* = -A$ is not a necessary condition for the vanishing of the inner product (XA,X) when $1 < k < n-1$. Certainly, it is not suggested by the proof of (6.2.11) that $A^* = -A$ is a necessary condition for the identical vanishing of (XA,X) when $1 < k < n-1$, for there we made use of the fact that the sum of two vectors in R_n is again a vector in R_n, a fact when $k = 1$ (or $k = n-1$) but not a fact, by (5.3.9), when $1 < k < n-1$, for k-vectors.

Suppose A is a real NxN matrix such that $A^* + A = Q$, where Q is some Grassmann matrix, and let X be any k-vector in the associated N-space of R_n. Then, by (4.1.1), (4.1.4), (4.1.14), and (5.7.6),

$$\begin{aligned}(XA,X) &= \tfrac{1}{2}\{(X,XA) + (XA,X)\}\\ &= \tfrac{1}{2}\{(XA^*,X) + (XA,X)\}\\ &= \tfrac{1}{2}(X[A^* + A],X)\\ &= \tfrac{1}{2}(XQ,X)\\ &= 0.\end{aligned}$$

This shows that $A^* + A = Q$ is a sufficient condition for

(XA,X) to vanish identically for k-vectors. In the cases $k = 1$ and $k = n-1$, the set of Grassmann matrices consists of only the $N \times N$ zero matrix; hence, in these cases the condition becomes $A^* + A = 0_n$, in agreement with (6.2.11). Therefore, we have the following theorem.

(6.2.12) Theorem. If A be the matrix of a linear transformation on the associated N-space of R_n, then a sufficient condition that the inner product of XA with X vanish for every k-vector in R_N is that there exist a Grassmann matrix Q such that

$$A^* + A = Q.$$

It will be shown later that the condition in (6.2.12) is also a necessary condition for the special case in which $k = 2$ and $n = 4$. It seems reasonable to expect that the condition is necessary in all cases. Hence we make the following conjecture.

(6.2.13) Conjecture. If A be the matrix of a linear transformation on the associated N-space of R_n, then a necessary condition that the inner product of XA with X vanish for every k-vector in R_N is that there exist a Grassmann matrix Q such that

$$A^* + A = Q.$$

Any subsequent theorems which depend on (6.2.13) will be marked with an asterisk.

Let $A = [a^i_j]$ be the matrix of a linear transformation over R_N such that

$$(XA,X) = 0$$

for every k-vector in R_N. Then we have

$$\begin{aligned} 0 &= (XA,X) + (XA,X) \\ &= (X,XA^*) + (XA,X) \\ &= (XA^*,X) + (XA,X) \\ &= (X[A^* + A],X) \end{aligned}$$

for every k-vector X in R_N. Therefore it is necessary that the inner product of $X[A^* + A]$ with X vanish identically in order that the inner product of XA with X vanish identically. If we set $B = A^* + A$, then $b^i_j = a^j_i + a^i_j$ and $B^* = B$. Thus we need only consider symmetric matrices. Suppose, therefore, $(XB,X) = 0$ for every k-vector in R_N. Since, by (5.4.3), every k-cell is equivalent to some k-cell with initial vertex at the origin, we need only consider the identity

$$\left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^k \end{pmatrix} B \;\middle|\; \begin{matrix} \theta \\ x^1 \\ \vdots \\ x^k \end{matrix} \right] = 0.$$

If B is an $N \times N$ symmetric matrix such that this identity holds, we have, by (3.4.11), (4.2.2), (4.2.16), and (6.2.1),

$$0 = \left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^v + y \\ \vdots \\ x^k \end{pmatrix} B \,\middle|\, \begin{matrix} \theta \\ x^1 \\ \vdots \\ x^v + y \\ \vdots \\ x^k \end{matrix} \right]$$

$$= \left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^v \\ \vdots \\ x^k \end{pmatrix} B \,\middle|\, \begin{matrix} \theta \\ x^1 \\ \vdots \\ x^v \\ \vdots \\ x^k \end{matrix} \right] + \left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^v \\ \vdots \\ x^k \end{pmatrix} B \,\middle|\, \begin{matrix} \theta \\ x^1 \\ \vdots \\ y \\ \vdots \\ x^k \end{matrix} \right] + \left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ y \\ \vdots \\ x^k \end{pmatrix} B \,\middle|\, \begin{matrix} \theta \\ x^1 \\ \vdots \\ x^v \\ \vdots \\ x^k \end{matrix} \right] + \left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ y \\ \vdots \\ x^k \end{pmatrix} B \,\middle|\, \begin{matrix} \theta \\ x^1 \\ \vdots \\ y \\ \vdots \\ x^k \end{matrix} \right]$$

$$= 2 \left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^v \\ \vdots \\ x^k \end{pmatrix} B \,\middle|\, \begin{matrix} \theta \\ x^1 \\ \vdots \\ y \\ \vdots \\ x^k \end{matrix} \right], \text{ where } y \text{ is any vector in } R_n .$$

Therefore we have the following theorem.

(6.2.14) Theorem. If, for the matrix A of a linear transformation over R_N, the identity

$$\left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^k \end{pmatrix} A \,\middle|\, \begin{matrix} \theta \\ x^1 \\ \vdots \\ x^k \end{matrix} \right] = 0$$

holds for k-vectors, then so does the identity

$$\left[\begin{pmatrix} \theta \\ x^1 \\ \vdots \\ x^v \\ \vdots \\ x^k \end{pmatrix} \left[A^* + A\right] \,\middle|\, \begin{matrix} \theta \\ x^1 \\ \vdots \\ y \\ \vdots \\ x^k \end{matrix} \right] = 0,$$

where y is an arbitrary vector in R_n, hold for $v = 1, \dots, k$.

In particular, let $x^u = e^{i_u}$ and $y = e^t$, where e^r is the r-th basis vector for R_n, in (6.2.14). Furthermore, let $[A^*+A]^{i_1\cdots i_k}_{j_1\cdots j_k}$ denote the i,j-th element of A^*+A when $(i_1,\ldots,i_k)$ and $(j_1,\ldots,j_k)$ are even (or odd) permutations of the i-th and j-th terms respectively of the sequence of the k-tuples of the first n natural numbers under lexicographical ordering, let $[A^*+A]^{i_1\cdots i_k}_{j_1\cdots j_k}$ denote the negative of the i,j-th term of A^*+A when $(i_1,\ldots,i_k)$ is an even (or odd) and $(j_1,\ldots,j_k)$ is an odd (or even) permutation, and let $[A^*+A]^{i_1\cdots i_k}_{j_1\cdots j_k}$ have the value zero when there are fewer than k distinct integers in either the set $\{i_1,\ldots,i_k\}$ or $\{j_1,\ldots,j_k\}$. Then, since the k-vectors determined by $((\theta,e^{i_1},\ldots,e^{i_k}))$ and $((\theta,e^{i_1},\ldots,e^{i_{v-1}},e^t,e^{i_{v+1}},\ldots,e^{i_k}))$ each have only one non-zero component, we have, by (6.2.14),

$$E^i_i\,[A]^i_{(i_v)_t}\,E^{(i_v)_t}_{(i_v)_t} = 0 \quad ,$$

where $(i_1,\ldots,i_{v-1},t,i_{v+1},\ldots,i_k)$ is some permutation of the $(i_v)_t$-th k-tuple of the sequence of the k-tuples of the first n natural numbers under lexicographical ordering whenever $t \neq i_1,\ldots,i_{v-1},i_{v+1},\ldots,i_k$. Let us suppose, therefore, that $1 \leqq i_1 < \ldots < i_k \leqq n$ and $t \neq i_1,\ldots,i_{v-1}, i_{v+1},\ldots,i_k$. Then $E_i^i \neq 0$ and $E_{(i_v)_t}^{(i_v)_t} \neq 0$, and hence

$$[A]^i_{(i_v)_t} = 0 .$$

We have proved the following theorem.

(6.2.15) <u>Theorem.</u> Let A be the matrix of a linear transformation over R_N ; let $[A^*+A]^i_j$ denote the i,j-th element of the matrix obtained by adding to A its transpose; and let $[A^*+A]^{i_1\ldots i_k}_{j_1\ldots j_k}$ have the value $[A^*+A]^i_j$ when $(i_1,\ldots,i_k)$ and $(j_1,\ldots,j_k)$ are even (or odd) permutations of the i-th and j-th terms respectively of the sequence of the k-tuples of the first n natural numbers under lexicographical ordering, the value $-[A^*+A]^i_j$ when $(i_1,\ldots,i_k)$ is an even (or odd) and $(j_1,\ldots,j_k)$ is an odd (or even) permutation, and the value zero when there are less than k distinct integers in either the set $\{i_1,\ldots,i_k\}$ or $\{j_1,\ldots,j_k\}$. Then, if the inner product of XA with X vanishes for every k-vector X in R_N , it is necessary that

$$[A^*+A]^{i_1\ldots i_k}_{i_1\ldots i_{v-1}t\ i_{v+1}\ldots i_k} = [A^*+A]^{i_1\ldots i_{v-1}t\ i_{v+1}\ldots i_k}_{i_1\ldots i_k} = 0$$

for $i = 1, \ldots , N$ and $t = 1, \ldots , n$.

Now we are ready to prove (6.2.13) for the case in which $k = 2$ and $n = 4$. Let A be any 6x6 real matrix such that $(XA,X) = 0$ for every 2-vector in R_6. Let

$$a_i^j + a_j^i = \left[A^* + A\right]_{j_1 j_2}^{i_1 i_2}, \quad 1 \leqq i_1 < i_2 \leqq 6, \quad 1 \leqq j_1 < j_2 \leqq 6.$$

Then we have, by (6.2.15), for lexicographical ordering,

$$A^* + A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & b_6^1 \\ 0 & 0 & 0 & 0 & b_5^2 & 0 \\ 0 & 0 & 0 & b_4^3 & 0 & 0 \\ 0 & 0 & b_3^4 & 0 & 0 & 0 \\ 0 & b_2^5 & 0 & 0 & 0 & 0 \\ b_1^6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $b_j^i = a_i^j + a_j^i$. Hence, for every 2-vector $(\xi_1, \ldots, \xi_6)$, we must have

$$\begin{bmatrix} \xi_1 \cdots \xi_6 \end{bmatrix} \begin{bmatrix} 0 & & b_6^1 \\ & & \\ b_1^6 & & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_6 \end{bmatrix} = 0 \quad ,$$

that is, since $b_j^i = b_i^j$,

$$b_6^1 \xi_1 \xi_6 + b_5^2 \xi_2 \xi_5 + b_4^3 \xi_3 \xi_4 = 0.$$

But, by (5.6.3), $\xi_1 \xi_6 = \xi_2 \xi_5 - \xi_3 \xi_4$; hence, for every 2-vector ξ, we must have

$$(b_6^1 + b_5^2)\ \xi_2 \xi_5 = (b_6^1 - b_4^3)\ \xi_3 \xi_4.$$

Setting $b_6^1 = c$, we have, then,

$b_1^6 = b_6^1 = c, \quad b_2^5 = b_5^2 = -c,$ and $\quad b_3^4 = b_4^3 = c.$

It follows therefore that

$$A^* + A = cQ,$$

where Q is the special Grassmann matrix in (5.6.4). This proves (6.2.13) for the case in which $k = 2$ and $n = 4$. Moreover, it follows from (5.7.4) and (6.2.11) that (6.2.13) holds for $k = 1$ and $k = n-1$. Hence we have the next theorem.

(6.2.16) Theorem. (6.2.13) is true whenever $k = 1$, $k = n-1$, or $k = 2$ and $n = 4$.

6.3. Transformations Which Preserve Inner Products and Norms. If U is a unitary matrix then $UU^* = U^*U = I$ and $(XU, YU) = (XUU^*, Y) = (X, Y)$ for all vectors in C_N, and therefore, in particular, for all k-vectors in the associated N-space of C_n. Conversely, if $(XU, YU) = (X, Y)$ for all k-vectors, then $(X[UU^* - I], Y) = 0$ for all k-vectors. Therefore, by (6.2.2), $UU^* = I$ and, consequently, as noted above, $(XU, YU) = (X, Y)$ for all vectors in C_N. Thus, we have proved the following theorem.

(6.3.1) Theorem. A linear transformation on C_N preserves inner products if, and only if, it preserves inner products of k-vectors in the associated N-space of C_n; a necessary and sufficient condition for this being that its matrix be unitary.

Since a real unitary matrix is an orthogonal matrix and since (6.2.2) holds for R_N as well as C_N, we have the next theorem.

(6.3.2) Theorem. A linear transformation on R_N preserves inner products if, and only if, it preserves inner products of k-vectors in the associated N-space of R_n and it does this if, and only if, its matrix is orthogonal.

Now $A^{(k)}$ is a unitary matrix if, and only if,

$$A^{(k)} \left[A^{(k)}\right]^* = \left[A^{(k)}\right]^* A^{(k)} = I_N = \left[I_n\right]^{(k)} ,$$

which is true if, and only if,

$$\left[A\ A^*\right]^{(k)} = \left[A^*\ A\right]^{(k)} = I^{(k)},$$

and this is true when $0 < k < n$, by (3) of (1.4.11) if, and only if, $AA^* = A^* A = \omega I$, where ω is a k-th root of unity, since A is clearly non-singular. But this is true if, and only if,

$$\sum_{\nu=1}^{n} a^i_\nu \bar{a}^j_\nu = \omega \delta^i_j .$$

Hence we must have

$$\sum_{\nu=1}^{n} a^i_\nu \bar{a}^i_\nu = \omega .$$

Since the left member of this equation is obviously a positive real number, $\omega = 1$. This proves the following theorem, since the argument is the same in the real case.

(6.3.3) <u>Theorem.</u> If A be a complex (or real) $n \times n$ matrix, then $A^{(k)}$ is unitary (or orthogonal) if, and only if, A is unitary (or orthogonal).

In order that a linear transformation with matrix A preserve generalized inner products of k-cells it is necessary and sufficient, by (6.1.2) and (5.4.7), that

$$(XA^{(k)}, YA^{(k)}) = (X, Y)$$

for all k-vectors in the associated N-space of C_n (or R_n). This is true, by (6.3.1) (or (6.3.2)) if, and only if, $A^{(k)}$ is unitary (or orthogonal). Hence we have the following theorem, in view of (6.3.3).

(6.3.4) <u>Theorem.</u> A linear transformation over the base space C_n (or R_n) preserves inner products of k-cells if, and only if, its matrix is a unitary (or an orthogonal) matrix.

Let A be the matrix of a linear transformation on C_N. Then a necessary and sufficient condition that this linear transformation leave the lengths of k-vectors in the associated N-space of C_n unchanged is that,

$$(XA, XA) = (X, X),$$

for all k-vectors X in C_N. This is true if, and only if,

$$(X[AA^* - I], X) = 0$$

for all k-vectors X in C_N. But this, in view of (6.2.7), is true if, and only if, $(X[AA^* - I], Y) = 0$ for every pair X, Y of k-vectors in C_N. Clearly, this is true if, and only if, $(XA, YA) = (X, Y)$ for every pair X, Y of k-vectors in C_N. Whereas, by (6.3.1), this is true if, and only if, A is unitary. Since A being unitary is also a necessary and sufficient condition that the linear transformation preserve the lengths of all vectors in C_N, we have the following theorem.

(6.3.5) <u>Theorem.</u> A linear transformation preserves lengths in C_N if, and only if, it preserves lengths of k-vectors in the associated N-space of C_n; a necessary and sufficient condition for this being that its matrix be unitary.

Since, by (5.4.7) and (6.1.2), a linear transformation over C_n with matrix A preserves volumes (norms) of k-cells in C_n if, and only if, the linear transformation whose matrix is the k-th compound of A preserves lengths of k-vectors in the associated N-space of C_n and since, by (6.3.3), $A^{(k)}$ is unitary if, and only if, A is unitary, the next theorem is a corollary to (6.3.5).

(6.3.6) <u>Theorem.</u> A linear transformation on C_n preserves volumes (norms) of k-cells in C_n if, and only if, its matrix is unitary.

Let A be a real $N \times N$ matrix, and suppose that there exists a Grassmann matrix Q such that $AA^* = Q + I_N$. Then, for every k-vector in R_N, we have

$$(X, X) = (XQ, X) + (XI, X)$$

$$= (X[Q + I], X)$$

$$= (XAA^*, X)$$

$$= (XA, XA).$$

It follows, therefore, that the next theorem is true.

(6.3.7) Theorem. A real linear transformation with matrix A preserves lengths of k-vectors in R_N if there exists a Grassmann matrix Q such that

$$A\,A^* = Q + I\,,$$

where I is the NxN unit matrix.

Conversely, suppose A to be such that

$$(XA, XA) = (X, X)$$

for all k-vectors in R_N. Then

$$(XAA^*, X) = (XI, X)\,,$$

that is,

$$(X[AA^*-I]\,, X) = 0$$

for all k-vectors in R_N. Thus, if A is real, it follows from (6.2.13) that there exists a Grassmann matrix Q_1 such that

$$[AA^*-I]^* + [AA^*-I] = Q_1\,.$$

Hence $AA^* = \frac{1}{2}Q_1 + I$. Since $\frac{1}{2}Q_1 = Q$ is also a Grassmann matrix whenever Q_1 is one, we have the next theorem.

$(6.3.8)^*$ Theorem. A real linear transformation with matrix A preserves lengths of k-vectors in R_N only if there exists a Grassmann matrix Q such that

$$A A^* = Q + I ,$$

where I is the $N\times N$ unit matrix.

Since, by (5.4.7) and (6.1.2), a linear transformation over R_N with matrix A preserves volumes (norms) of k-cells in R_n if, and only if, the linear transformation whose matrix is the k-th compound of A preserves lengths of k-vectors in the associated N-space of R_n, the next two theorems follow as corollaries to (6.3.7) and (6.3.8) respectively.

(6.3.9) Theorem. A real linear transformation with matrix A preserves volumes (norms) of k-cells in R_n if there exists a Grassmann matrix Q such that

$$[AA^*]^{(k)} = Q + I.$$

$(6.3.10)^*$ Theorem. A real linear transformation with matrix A preserves volumes (norms) of k-cells in R_n only if there exists a Grassmann matrix Q such that

$$[AA^*]^{(k)} = Q + I.$$

6.4. Transformations Which Preserve the Set of k-Vectors. Since, by (5.3.9), the sum of two k-vectors is not, in general, a k-vector, it is clear that only a very special kind of linear transformation will carry the set of k-vectors into itself. We already know of an entire class of such transformations, namely, those whose matrices are k-th compounds of $n \times n$ matrices. Now it is natural to ask whether there are any other linear transformations with this property. Therefore we seek necessary and sufficient conditions in order that a linear transformation carry k-vectors into k-vectors and non-k-vectors into non-k-vectors. In other words, we seek necessary and sufficient conditions in order that a linear transformation preserve the geometric configuration, in the associated N-space, consisting of the set $\{X\}$ of k-vectors. Notationally, we seek necessary and sufficient conditions that, for a linear transformation with matrix A, the set identity $\{X\} = \{XA\}$ should hold. It follows from (5.7.6) that this identity holds if, and only if, the set of solutions of the set

$$X\{Q^{n,k}\}X' = 0$$

of matrix equations coincides with the set of solutions of the set

$$XA\{Q^{n,k}\}A'X' = 0 \quad ,$$

where the prime denotes "transposed". Hence we have the following theorem.

(6.4.1) Theorem. A necessary and sufficient condition that a linear transformation, with matrix A, on the associated N-space carry the set of k-vectors into itself is that

$$A\{Q^{n,k}\}A' = \{Q^{n,k}\} ,$$

that is, for every Grassmann matrix $Q^{n,k}$, the matrix $A\,Q^{n,k}A'$ is a Grassmann matrix and the set $\{A\,Q^{n,k}A'\}$ coincides with the set $\{Q^{n,k}\}$ of Grassmann matrices.

It is clear that any transformation which satisfies the condition of (6.4.1) must be non-singular.

For every non-singular $n \times n$ matrix A, the mapping $y = xA$ determines an automorphism of $L_n(K)$ and therefore, since then $((y^0,\ldots,y^k)) = ((x^0A,\ldots,x^kA))$, an automorphism of the associated N-space. Consequently, because of (6.1.2), the mapping

$$Y = XA^{(k)}$$

determines an automorphism of the set of k-vectors $X,Y,\ldots$ in the associated N-space. Hence it follows from (6.4.1) that the next theorem is true.

(6.4.2) Theorem. If A be any non-singular $n \times n$ matrix then

$$A^{(k)}\;\{Q^{n,k}\}\,[A^{(k)}]' = \{Q^{n,k}\} ,$$

where $\{Q^{n,k}\}$ denotes the set of Grassmann matrices.

Applying (6.4.1) to (5.8.17) and observing that $H_{n,k}$ is symmetric, we have the next theorem.

(6.4.3) <u>Theorem.</u> Whenever $n = 2k$, $H_{n,k}$, as well as $H'_{n,k}$, is the matrix of a linear transformation, on the associated N-space, which carries the set of k-vectors into itself.

Applying (6.4.1) to (5.8.18), we have, in view of (1.8.5), the next theorem.

(6.4.4) <u>Theorem.</u> Whenever $n = 2k$, $G_{n,k}$, as well as $G'_{n,k}$, is the matrix of a linear transformation, on the associated N-space, which carries the set of k-vectors into itself.

Now it is clear that we have in (6.4.2) a necessary condition for a matrix to be the k-th compound of an nxn matrix. If it were also a sufficient condition, we would have in it a very simple method of characterizing compound matrices, namely, as those which preserve the set of Grassmann matrices or, in other words, as the matrices of those linear transformations on $L_N(K)$ which preserve the set of k-vectors. Unfortunately, this is not always the case. Indeed, in the very simplest instance, namely, when $n = 4$ and $k = 2$, there are matrices which preserve the Grassmann matrices but are not second compounds. To wit, all matrices of the form $c\,H_{4,2}$ and all matrices of the form $c\, G_{4,2}$, where c is any non-zero element of the ground field K. That such matrices do preserve the set of Grassmann matrices follows immediately from (6.4.3) and (6.4.4). We shall prove that no matrix of the form $c\, G_{4,2}$ can be a second compound. A similar argument will show that no matrix of the form $c\, H_{4,2}$ can be a second compound. To show that no matrix of the form $c\, G_{4,2}$, where c is any non-zero element of K, is the second compound of a 4x4 matrix over K, we need merely show that $G_{4,2}$ cannot be a second compound. To this end, let us suppose that there does exist a matrix

$$A = \left[a^i_j\right] , \qquad i,j = 1,2,3,4,$$

such that

$$A^{(2)} = G_{4,2} .$$

Then if, taking the order to be lexicographical, we identify the i,j-th elements of the two matrices for $i,j = 1,1;\ 1,2;\ 1,5;\ 1,6;\ 2,4;\ 2,5;\ 3,4;\ 4,2;\ 4,3;$ and $5,2$, we see that the following ten relations must hold simultaneously, namely:

$$a^1_1\, a^2_2 - a^1_2\, a^2_1 = 0 ,$$

$$a^1_1\, a^2_3 - a^1_3\, a^2_1 = 0 ,$$

$$a^1_2\, a^2_4 - a^1_4\, a^2_2 = 0 ,$$

$$a^1_3\, a^2_4 - a^1_4\, a^2_3 = 1 ,$$

$$a^1_2\, a^3_3 - a^1_3\, a^3_2 = 0\ ,$$

$$a^1_4\, a^3_2 - a^1_2\, a^3_4 = 1\ ,$$

$$a^1_2\, a^4_3 - a^1_3\, a^4_2 = 1\ ,$$

$$a^2_1\, a^3_3 - a^2_3\, a^3_1 = 0\ ,$$

$$a^2_1\, a^3_4 - a^2_4\, a^3_1 = 1\ ,$$

$$a^2_3\, a^4_1 - a^2_1\, a^4_3 = 1\ .$$

Now we shall show that the first four relations imply $a^1_2\, a^2_1 = 0$ whereas the second three and the third three imply respectively $a^1_2 \neq 0$ and $a^2_1 \neq 0$, which is absurd. To this end, let us multiply both members of the fourth relation by $a^1_2\, a^2_1$ to obtain

$$a^1_2\, a^2_4\, a^1_3\, a^2_1 - a^1_4\, a^2_3\, a^1_2\, a^2_1 = a^1_2\, a^2_1\ ,$$

in which we substitute $a^1_4\, a^2_2$ for $a^1_2\, a^2_4$, $a^1_1\, a^2_3$ for $a^1_3\, a^2_1$, and $a^1_1\, a^2_2$ for $a^1_2\, a^2_1$ in the second term, by virtue of the third, second, and first relations respectively, to obtain

$$a^1_4\, a^2_2\, a^1_1\, a^2_3 - a^1_4\, a^2_3\, a^1_1\, a^2_2 = a^1_2\, a^2_1\ .$$

This shows that $a^1_2\, a^2_1 = 0$.

On the other hand, if we multiply the sixth relation by the seventh relation, we have

$$1 = (a_2^1\, a_3^4 - a_3^1\, a_2^4)\,(a_4^1\, a_2^3 - a_2^1\, a_4^3)$$
$$= a_2^1\,(a_3^4\, a_4^1\, a_2^3 - a_3^4\, a_2^1\, a_4^3 + a_3^1\, a_2^4\, a_4^3) - a_3^1\, a_2^3\, a_4^1\, a_2^4;$$

substituting $a_2^1\, a_3^3$ for $a_3^1\, a_2^3$ in the last term, by virtue of the fifth relation, we obtain, upon factoring,

$$1 = a_2^1\,(a_3^4\, a_4^1\, a_2^3 - a_3^4\, a_2^1\, a_4^3 + a_3^1\, a_2^4\, a_4^3 - a_3^3\, a_4^1\, a_2^4)\ ,$$

which shows that $a_2^1 \neq 0$.

Similarly, if we multiply the ninth relation by the tenth and substitute $a_1^2\, a_3^3$ for $a_3^2\, a_1^3$ in the result, by virtue of the eighth relation, we obtain, upon factoring,

$$1 = a_1^2\,(a_4^3\, a_3^2\, a_1^4 - a_4^3\, a_1^2\, a_3^4 + a_3^4\, a_4^2\, a_1^3 - a_3^3\, a_1^4\, a_4^2)\ ,$$

which shows that $a_1^2 \neq 0$.

Thus the assumption that there exists a matrix A such that

$$A^{(2)} = G_{4,2}$$

leads us to the absurd conclusion that, simultaneously,

$$a_2^1\, a_1^2 = 0 \qquad \text{and} \qquad a_2^1\, a_1^2 \neq 0\ .$$

We summarize the conclusions to be drawn from the last several paragraphs in the next theorem.

(6.4.3) <u>Theorem.</u> If $n = 4$, $k = 2$, and c is any non-zero element of the ground field K, then the set of matrices of the form $cG_{4,2}$,

$$G_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

has the following properties:

(i) no matrix of the set is the second compound of any matrix whatsoever;

(ii) every matrix of the set preserves the set of Grassmann matrices, in other words, every matrix of the set is the matrix of a linear transformation on the associated N-space of $L_2(K)$ which preserves the set of 2-vectors.

A similar argument shows that the next theorem is also true.

(6.4.4) <u>Theorem.</u> If $n = 4$, $k = 2$, and c is any non-zero element of the ground field K, then the set of matrices of the form $c\,H_{4,2}$, where

$$H_{4,2} = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{bmatrix},$$

has the following properties:

(i) no matrix of the set is the second compound of any matrix whatsoever;

(ii) every matrix of the set is the matrix of a linear transformation on $L_N(K)$ which preserves the set of 2-vectors.

6.5. The Grassmann Group. The set of non-singular linear transformations which leaves unchanged a particular configuration in a linear space forms a group. Hence we make the following definition.

(6.5.1) Definition. The set consisting of those non-singular linear transformations on the associated N-space of $L_n(K)$ which transform the set of k-vectors into itself will be called the Grassmann group.

Now the next theorem follows from (6.4.1).

(6.5.2) Theorem. A non-singular matrix A is the matrix of a transformation in the Grassmann group if, and only if, for every Grassmann matrix $Q^{n,k}$, the matrix $A\, Q^{n,k}\, A'$ is again a Grassmann matrix.

Since the set of matrices which are k-th compounds of non-singular matrices of order n, for fixed n and k, is a group, the next theorem is true.

(6.5.3) <u>Theorem.</u> The set of linear transformations whose matrices are k-th compounds of non-singular matrices of order n is a subgroup of the Grassmann Group.

It follows from (6.4.3) when $n = 4$, $k = 2$ and more generally from the considerations in the next chapter that the next theorem is true.

(6.5.4) <u>Theorem.</u> When $1 < k$ and $n = 2k$, the set of linear transformations whose matrices are k-th compounds of non-singular matrices of order n is a <u>proper</u> subgroup of the Grassmann group.

For then, the Grassmann group contains the matrices $c\,H_{2k,k}$ and $c\,G_{2k,k}$, where c is any non-zero element of the ground field K, which are not k-th compounds whenever $1 < k < n-1$.

Chapter 7

The Matrix Equation $X^{(k)} = B$

Let k be a given non-negative integer and let B be a given matrix over the field of complex numbers. In this chapter, we seek necessary and sufficient conditions that there exist a matrix X whose k-th compound is B.

<u>7.1. Preliminaries.</u> It is clear from the considerations of Section 1.2 that there can be no $m \times n$ matrix X whose k-th compound is a given matrix B if B is not a $C(m,k) \times C(n,k)$ matrix. Moreover, by (1.4.4), we see that B must have rank $C(r,k)$, for some r, or there can be no solution. We remark that the problem is trivial when $k = 0$, in view of (1.2.1), and also when $m = n = k$, that is, when B has a single entry, for then any $k \times k$ matrix X whose determinant equals the single entry of B is a solution. Furthermore, when $k = 1$, there is always a solution, for then $X = B$ is the solution. We see, therefore, that we need consider only the cases in which $1 < k < n$ when we are considering square matrices.

Suppose, therefore, $0 < k < n$ and let A be any solution of the matrix equation

$$X^{(n-k)} = B \quad ,$$

where B is a given, non-singular matrix of order $N = C(n,k)$.

Then, by hypothesis, A is a non-singular matrix of order n and, by (1.4.9), we have

$$|B| = |A|^{C(n-1,k)}$$

that is,

$$|A| = |B|^{1/C(n-1,k)} .$$

But, by (1.9.4), if we set $G = G_{n,k} G_{N,1}$ and $p = (1-C(n-1,k)) / C(n-1,k)$, then

$$A^{(k)} = |A|^{1-C(n-1,k)} \; G \left[A^{(n-k)}\right]^{(N-1)} \; G'$$

$$= |B|^{p} \; G \; B^{(N-1)} \; G' .$$

Whence A is also a solution of the matrix equation

$$X^{(k)} = |B|^{p} \; G \; B^{(N-1)} \; G' .$$

Conversely, suppose A is any solution of the matrix equation

$$X^{(k)} = |B|^{p} \; G \; B^{(N-1)} \; G' .$$

Then, since $G = G_{n,k} G_{N,1}$, we have, by (1.8.2) and (1.8.4),

$$G'_{n,k} \; A^{(k)} \; G_{n,k} = |B|^{p} \; G_{N,1} \; B^{(N-1)} \; G'_{N,1}$$

$$= |B|^{p} \; G'_{N,N-1} \; B^{N-1} \; G_{N,N-1} .$$

Taking determinants of both sides and using (1.4.9), we obtain, after simplifying, since $N = C(n,k)$ and $p = (1-C(n-1,k))/C(n-1,k)$,

$$|A|^{C(n-1,k)} = |B| .$$

Hence

$$|B|\, G'_{n,k}\, A^{(k)}\, G_{n,k} = |A|\, G'_{N,1}\, B^{(N-1)}\, G_{N,1} ,$$

and so, by (1.9.1),

$$|B| \operatorname{adj}^{(n-k)} A' = |A| \operatorname{adj} B' .$$

Taking the transpose of both sides and applying (1.4.7), we obtain

$$|B|\,|A| \left[A^{(n-k)}\right]^{-1} = |A|\,|B|\, B^{-1} .$$

Cancelling out the common factor and taking inverses, we obtain

$$A^{(n-k)} = B ,$$

which shows that A is also a solution of the equation

$$X^{(n-k)} = B .$$

Therefore we have proved the following theorem for square matrices.

(7.1.1) <u>Theorem.</u> Let B be a given non-singular matrix of order N, $N = C(n,k)$ and $0 < k < n$. Then any solution of

$$X^{(n-k)} = B$$

is also a solution of

$$X^{(k)} = \|B\|^{p} \, G \, B^{(N-1)} \, G' \; ,$$

where $G = G_{n,k} G_{N,1}$ and $p = (1-C(n-1,k)) / C(n-1,k)$, and conversely.

Since any solution of $X^{(k)} = \|B\|^{p} \, G \, B^{(N-1)} \, G'$ is $\|B\|^{p/k}$ times a solution of $X^{(k)} = G \, B^{(N-1)} \, G'$, it follows that the next theorem is true also.

(7.1.2) <u>Theorem.</u> Let B be a given non-singular matrix of order N, $N = C(n,k)$ and $0 < k < n$, and let $G = G_{n,k} \, G_{N,1}$. Then the matrix equation

$$X^{(n-k)} = B$$

has a solution if, and only if, the equation

$$X^{(k)} = G \, B^{(N-1)} \, G'$$

has a solution.

The next corollary follows immediately from (7.1.2).

(7.1.3) <u>Corollary.</u> Let $N = C(n,k)$, $0 < k < n$, and $G = G_{n,k} \, G_{N,1}$. Then a non-singular matrix B of order N is an $(n-k)$-th compound if, and only if, $G \, B^{(N-1)} \, G'$ is a k-th compound.

Now, using (1.8.2) and (1.8.4), we have

$$G\,B^{(N-1)}\,G' = G_{n,k}\,G_{N,1}\,B^{(N-1)}\,G'_{N,1}\,G'_{n,k}$$

$$= G'_{n,n-k}\,G'_{N,N-1}\,B^{(N-1)}\,G_{N,N-1}\,G_{n,n-k}$$

$$= G^{-1}_{n,n-k}\,G^{-1}_{N,N-1}\,B^{(N-1)}\,G_{N,N-1}\,G_{n,n-k}$$

$$= \left[G_{N,N-1}\,G_{n,n-k}\right]^{-1} B^{(N-1)} \left[G_{N,N-1}\,G_{n,n-k}\right] .$$

Hence the next corollary follows from (7.1.3).

(7.1.4) <u>Corollary.</u> A non-singular matrix of order N, $N = C(n,k)$ and $0 < k < n$, is an $(n-k)$-th compound if, and only if, the transform of its $(N-1)$-th compound by $G_{N,N-1}\,G_{n,n-k}$ is a k-th compound.

<u>7.2. Solution of $X^{(k)} = B$ When $m = n = k+1$.</u> It follows from the fact that $X^{(1)} = B$ always has a solution and (7.1.4) that $X^{(k)} = B$ has a solution when $n = k+1$ and B is any non-singular matrix of order n. If $n = k+1$, then $k = n-1$ and $N = C(n,k) = C(n,n-1) = C(n,1) = n$. Let B be any non-singular matrix of order n. If

$$X^{(n-1)} = B ,$$

then, by (7.1.1), and (1.7.5), we have

$$X^{(1)} = |B|^{p}\,G\,B^{(N-1)}\,G'$$

$$= |B|^{\frac{2-n}{n-1}} G^{2}_{n,1} B^{(n-1)} \left[G'_{n,1}\right]^{2}$$

$$= |B|^{\frac{2-n}{n-1}} B^{(n-1)}$$

$$= |B|^{\frac{1-k}{k}} B^{(k)} \quad .$$

If a be any number such that $a^k |B|^{k-1} = 1$, then it follows that $X = a B^{(k)}$ is a solution of the equation $X^{(k)} = B$ whenever B is any non-singular matrix of order $n = k+1$. It follows from (1.4.11) that there are precisely k solutions, namely,

$$X = \omega_v a B^{(k)} , \qquad v = 1,\ldots,k,$$

where $\omega_1,\ldots,\omega_k$ are the k k-th roots of unity. Hence we have the following theorem.

(7.2.1) <u>Theorem.</u> If B be any non-singular matrix of order $k+1$, then the matrix equation

$$X^{(k)} = B$$

is always solvable, there being precisely k solutions,

$$X = \omega_v a B^{(k)} , \qquad v = 1,\ldots,k,$$

where the numbers $\omega_1,\ldots,\omega_k$ are the k k-th roots of unity and a is any number such that

$$a^k |B|^{k-1} = 1 \quad .$$

The following corollary is an obvious consequence of (7.2.1)

(7.2.2) <u>Corollary.</u> When B is a real, non-singular matrix of order $k+1$, the matrix equation

$$X^{(k)} = B$$

has no real solution, one real solution, or two real solutions--one being the negative of the other--according as k is an even integer and $|B| < 0$, k is an odd integer, or k is an even integer and $|B| > 0$.

It should be noted that (7.2.1) is also a consequence of (2.3.8). For example, when $k = 2$ and $n = k+1 = 3$, we have, using (2.3.8),

$$\left[A^{(2)}\right]^{(2)} = \left[\begin{bmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{bmatrix}^{(2)}\right]^{(2)}$$

$$= \begin{bmatrix} \begin{vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix} & \begin{vmatrix} a_1^1 & a_3^1 \\ a_1^2 & a_3^2 \end{vmatrix} & \begin{vmatrix} a_2^1 & a_3^1 \\ a_2^2 & a_3^2 \end{vmatrix} \\ \begin{vmatrix} a_1^1 & a_2^1 \\ a_1^3 & a_2^3 \end{vmatrix} & \begin{vmatrix} a_1^1 & a_3^1 \\ a_1^3 & a_3^3 \end{vmatrix} & \begin{vmatrix} a_2^1 & a_3^1 \\ a_2^3 & a_3^3 \end{vmatrix} \\ \begin{vmatrix} a_1^2 & a_2^2 \\ a_1^3 & a_2^3 \end{vmatrix} & \begin{vmatrix} a_1^2 & a_3^2 \\ a_1^3 & a_3^3 \end{vmatrix} & \begin{vmatrix} a_2^2 & a_3^2 \\ a_2^3 & a_3^3 \end{vmatrix} \end{bmatrix}^{(2)}$$

$$
\begin{bmatrix}
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{vmatrix} & \begin{vmatrix} a^1_1 & a^1_3 \\ a^2_1 & a^2_3 \end{vmatrix} \\ \begin{vmatrix} a^1_1 & a^1_2 \\ a^3_1 & a^3_2 \end{vmatrix} & \begin{vmatrix} a^1_1 & a^1_3 \\ a^3_1 & a^3_3 \end{vmatrix} \end{Vmatrix} &
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{vmatrix} & \begin{vmatrix} a^1_2 & a^1_3 \\ a^2_2 & a^2_3 \end{vmatrix} \\ \begin{vmatrix} a^1_1 & a^1_2 \\ a^3_1 & a^3_2 \end{vmatrix} & \begin{vmatrix} a^1_2 & a^1_3 \\ a^3_2 & a^3_3 \end{vmatrix} \end{Vmatrix} &
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_3 \\ a^2_1 & a^2_3 \end{vmatrix} & \begin{vmatrix} a^1_2 & a^1_3 \\ a^2_2 & a^2_3 \end{vmatrix} \\ \begin{vmatrix} a^1_1 & a^1_3 \\ a^3_1 & a^3_3 \end{vmatrix} & \begin{vmatrix} a^1_2 & a^1_3 \\ a^3_2 & a^3_3 \end{vmatrix} \end{Vmatrix} \\
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{vmatrix} & \begin{vmatrix} a^1_1 & a^1_3 \\ a^2_1 & a^2_3 \end{vmatrix} \\ \begin{vmatrix} a^2_1 & a^2_2 \\ a^3_1 & a^3_2 \end{vmatrix} & \begin{vmatrix} a^2_1 & a^2_3 \\ a^3_1 & a^3_3 \end{vmatrix} \end{Vmatrix} &
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{vmatrix} & \begin{vmatrix} a^1_2 & a^1_3 \\ a^2_2 & a^2_3 \end{vmatrix} \\ \begin{vmatrix} a^2_1 & a^2_2 \\ a^3_1 & a^3_2 \end{vmatrix} & \begin{vmatrix} a^2_2 & a^2_3 \\ a^3_2 & a^3_3 \end{vmatrix} \end{Vmatrix} &
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_3 \\ a^2_1 & a^2_3 \end{vmatrix} & \begin{vmatrix} a^1_2 & a^1_3 \\ a^2_2 & a^2_3 \end{vmatrix} \\ \begin{vmatrix} a^2_1 & a^2_3 \\ a^3_1 & a^3_3 \end{vmatrix} & \begin{vmatrix} a^2_2 & a^2_3 \\ a^3_2 & a^3_3 \end{vmatrix} \end{Vmatrix} \\
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_2 \\ a^3_1 & a^3_2 \end{vmatrix} & \begin{vmatrix} a^1_1 & a^1_3 \\ a^3_1 & a^3_3 \end{vmatrix} \\ \begin{vmatrix} a^2_1 & a^2_2 \\ a^3_1 & a^3_2 \end{vmatrix} & \begin{vmatrix} a^2_1 & a^2_3 \\ a^3_1 & a^3_3 \end{vmatrix} \end{Vmatrix} &
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_2 \\ a^3_1 & a^3_2 \end{vmatrix} & \begin{vmatrix} a^1_2 & a^1_3 \\ a^3_2 & a^3_3 \end{vmatrix} \\ \begin{vmatrix} a^2_1 & a^2_2 \\ a^3_1 & a^3_2 \end{vmatrix} & \begin{vmatrix} a^2_2 & a^2_3 \\ a^3_2 & a^3_3 \end{vmatrix} \end{Vmatrix} &
\begin{Vmatrix} \begin{vmatrix} a^1_1 & a^1_3 \\ a^3_1 & a^3_3 \end{vmatrix} & \begin{vmatrix} a^1_2 & a^1_3 \\ a^3_2 & a^3_3 \end{vmatrix} \\ \begin{vmatrix} a^2_1 & a^2_3 \\ a^3_1 & a^3_3 \end{vmatrix} & \begin{vmatrix} a^2_2 & a^2_3 \\ a^3_2 & a^3_3 \end{vmatrix} \end{Vmatrix}
\end{bmatrix}
$$

$$
= \begin{bmatrix} a^1_1 \cdot |A| & a^1_2 \cdot |A| & a^1_3 \cdot |A| \\ a^2_1 \cdot |A| & a^2_2 \cdot |A| & a^2_3 \cdot |A| \\ a^3_1 \cdot |A| & a^3_2 \cdot |A| & a^3_3 \cdot |A| \end{bmatrix}
$$

$$
= |A| \cdot A .
$$

Setting $B = A^{(2)}$, we have $|A|^2 = |B|$; and hence, if A is non-singular,

$$
A = \pm \frac{1}{\sqrt{|B|}} \cdot B^{(2)} .
$$

Let B be any square matrix of order $k+1$ and rank 1. Then B has at least one non-zero element, say $b^i_j \neq 0$, and each of the k remaining rows (and columns) is a linear combination of the i-th row (and j-th column). Consequently there exist non-singular matrices C and D of order $k+1$ such that $|C| = |D| = 1$ and the matrix CBD has b^i_j in its upper, left corner and zeros elsewhere. For let E_r $(r \neq i)$ be the elementary matrix obtained by performing that elementary row operation upon the identity matrix of order $k+1$ which will reduce the r-th row of B to a row of zeros, let E_i be the elementary matrix obtained by interchanging the first and i-th rows of the identity matrix, and E_0 be the elementary matrix obtained by interchanging any other pair of rows of I; then $C = E_0E_1\ldots E_{k+1}$ and, since $|E_0| = |E_i| = -1$ and $|E_r| = 1$ for $r = 1,\ldots,i-1, i+1,\ldots,k+1$, $|C| = 1$. D is obtained in a similar manner. Moreover, by (7.2.1) we know that, for $v = 1,2,\ldots,k$,

$$(7.2.3) \qquad \left[\omega_v C^{(k)}\right]^{(k)} = C \quad \text{and} \quad \left[\omega_v D^{(k)}\right]^{(k)} = D,$$

where the numbers $\omega_1,\ldots,\omega_k$ are the k k-th roots of unity. Now let Y be any square matrix of order $k+1$ such that the last row (and last column) is a linear combination of the k remaining rows (and columns) and $Y^{1\ldots k}_{1\ldots k} = b^i_j$, where $Y^{1\ldots k}_{1\ldots k}$ is the minor of order k in the upper, left corner of Y. Then clearly we have

$$Y^{(k)} = CBD,$$

and, since C and D are non-singular matrices,

$$C^{-1} Y^{(k)} D^{-1} = B \quad .$$

Hence it follows from (7.2.3) and (1.4.3) that

$$\left[\left[\omega_u C^{(k)}\right]^{-1} Y \left[\omega_v D^{(k)}\right]^{-1}\right]^{(k)} = B \quad .$$

Since $\omega_u^{k-1} \cdot \omega_v^{k-1}$ is again a k-th root of unity, we have proved the following theorem.

(7.2.4) <u>Theorem.</u> Let B be any matrix of order $k+1$ and rank 1, and let b be any non-zero element of B. Then there exist strictly unimodular matrices C and D such that the matrix CBD has b in its upper, left corner and zeros elsewhere. Moreover the matrix equation

$$X^{(k)} = B$$

is always solvable, there being at least k solutions,

$$X = \omega_v \left[C^{-1}\right]^{(k)} Y \left[D^{-1}\right]^{(k)} \quad ,$$

where the numbers $\omega_1, \ldots, \omega_k$ are the k k-th roots of unity and where Y is any matrix of order $k+1$ and rank k whose k-th order minor in the upper, left corner is equal to b.

Obviously, any matrix X of order $k+1$ and rank $r < k$ is a solution of

$$X^{(k)} = B$$

whenever the rank of B is zero. Hence we have the next theorem in view of (7.2.1), (7.2.4) and (1.4.4).

(7.2.5) <u>Theorem.</u> If B be a given matrix of order $k+1$ and rank r, then, for matrices over the field of complex numbers, the matrix equation

$$X^{(k)} = B$$

has precisely k distinct solutions when $r = k+1$, at least k distinct solutions when $r = 1$, infinitely many solutions when $r = 0$, and no solution when $1 < r < k+1$.

Finally, it follows from (7.2.2), the argument used to prove (7.2.4), and (1.4.4) that the next theorem is true.

(7.2.6) <u>Theorem.</u> If B be a given, real matrix of order $k+1$ and rank r, then the matrix equation

$$X^{(k)} = B$$

has precisely (at least) two real solutions--one being the negative of the other--when k is an even integer provided $|B| > 0$ (provided $r = 1$), precisely (at least) one real solution when k is an odd integer provided $|B| < 0$ (provided $r = 1$), infinitely many real solutions when $r = 0$, and no solution otherwise.

<u>7.3. Elements of X in Terms of Those of $X^{(k)}$.</u> It follows from (7.2.1) that if B be any non-singular matrix of order $k+1$, then

$$B = \left[a\, B^{(k)}\right]^{(k)} \quad \text{where} \quad a^k |B|^{k-1} = 1.$$

This expresses the elements of B in terms of those of the k-th compound of B. More generally, we shall show in this section that, if X be any $m \times n$ matrix of rank r, the elements of X can be expressed in terms of those of the k-th compound of X whenever m, n, and r are all greater than k.

If we denote the k-th order minor of an arbitrary matrix X which is common to the rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$ by $X^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ then it follows from the elementary properties of determinants that the following remark is true.

(7.3.1) <u>Remark.</u> $X^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ is skew-symmetric in both its upper suffixes and its lower suffixes; in other words, in order to change its sign it is sufficient to interchange either a pair of its upper or a pair of its lower suffixes, and its value is zero whenever two or more of its upper, or lower, suffixes are the same.

Since the cofactor of $x^{i_t}_{s_\sigma}$ in the determinant

$X^{i_1\cdots i_k}_{j_1\cdots j_{v-1}s_\sigma j_{v+1}\cdots j_k}$ is the cofactor $\tilde{x}^{i_t}_{j_v}$ of $x^{i_t}_{j_v}$ in $X^{i_1\cdots i_k}_{j_1\cdots j_k}$, it is clear that if we expand the former determinant in terms of the elements of the v-th columns, we have

$$\sum_{t=1}^{k} x^{i_t}_{s_\sigma} \tilde{x}^{i_t}_{j_v} = X^{i_1\cdots i_k}_{j_1\cdots j_{v-1}s_\sigma j_{v+1}\cdots j_k} .$$

Multiplying both members of this equation by $x^{i_u}_{j_v}$, summing over v, and reversing the order of the summation symbols on the left, we obtain

$$(7.3.2) \quad x^{i_u}_{s_\sigma} \cdot X^{i_1\cdots i_k}_{j_1\cdots j_k} = \sum_{\nu=1}^{k} x^{i_u}_{j_v} X^{i_1\cdots i_k}_{j_1\cdots j_{v-1}s_\sigma j_{v+1}\cdots j_k} .$$

Again, using the u-th row instead of the v-th column, we obtain, in an exactly like manner,

$$(7.3.3) \quad x^{r_\rho}_{j_v} X^{i_1\cdots i_k}_{j_1\cdots j_k} = \sum_{u=1}^{k} x^{i_u}_{j_v} X^{i_1\cdots i_{u-1}r_\rho i_{u+1}\cdots i_k}_{j_1\cdots j_k} .$$

For convenience, let us denote $X^{i_1\cdots i_k}_{j_1\cdots j_k}$ by X^i_j, $X^{i_1\cdots i_{u-1}r_\rho i_{u+1}\cdots i_k}_{j_1\cdots j_k}$ by $X^{ur_\rho}_j$, $X^{i_1\cdots i_k}_{j_1\cdots j_{v-1}s_\sigma j_{v+1}\cdots j_k}$ by $X^i_{vs_\sigma}$, and $X^{i_1\cdots i_{u-1}r_\rho i_{u+1}\cdots i_k}_{j_1\cdots j_{v-1}s_\sigma j_{v+1}\cdots j_k}$ by $X^{ur_\rho}_{vs_\sigma}$. Then it follows from (7.3.2) and (7.3.3) respectively that

$$(7.3.4)\qquad X^i_j \begin{bmatrix} x^{i_1}_{s_1} & \cdot & x^{i_1}_{s_k} \\ \cdot & \cdot & \cdot \\ x^{i_k}_{s_1} & \cdot & x^{i_k}_{s_k} \end{bmatrix} = \begin{bmatrix} x^{i_1}_{j_1} & \cdot & x^{i_1}_{j_k} \\ \cdot & \cdot & \cdot \\ x^{i_k}_{j_1} & \cdot & x^{i_k}_{j_k} \end{bmatrix} \begin{bmatrix} X^i_{1s_1} & \cdot & X^i_{1s_k} \\ \cdot & \cdot & \cdot \\ X^i_{ks_1} & \cdot & X^i_{ks_k} \end{bmatrix}$$

and

$$(7.3.5)\qquad X^i_j \begin{bmatrix} x^{r_1}_{j_1} & \cdot & x^{r_1}_{j_k} \\ \cdot & \cdot & \cdot \\ x^{r_k}_{j_1} & \cdot & x^{r_k}_{j_k} \end{bmatrix} = \begin{bmatrix} X^{1r_1}_j & \cdot & X^{kr_1}_j \\ \cdot & \cdot & \cdot \\ X^{1r_k}_j & \cdot & X^{kr_k}_j \end{bmatrix} \begin{bmatrix} x^{i_1}_{j_1} & \cdot & x^{i_1}_{j_k} \\ \cdot & \cdot & \cdot \\ x^{i_k}_{j_1} & \cdot & x^{i_k}_{j_k} \end{bmatrix} .$$

Taking determinants, we obtain from (7.3.4) and (7.3.5) respectively

$$(7.3.6)\qquad (X^i_j)^k\ X^i_s = x^i_j \begin{vmatrix} X^i_{1s_1} & \cdot & X^i_{1s_k} \\ \cdot & \cdot & \cdot \\ X^i_{ks_1} & \cdot & X^i_{ks_k} \end{vmatrix}$$

and

$$(7.3.7)\qquad (X^i_j)^k\ X^r_j = x^i_j \begin{vmatrix} X^{1r_1}_j & \cdot & X^{kr_1}_j \\ \cdot & \cdot & \cdot \\ X^{1r_k}_j & \cdot & X^{kr_k}_j \end{vmatrix} .$$

Now it is easy to see that

$$X^{ur_\rho}_{vs_\sigma} = x^{r_\rho}_{s_\sigma} \cdot \tilde{x}^{i_u}_{j_v} + \begin{vmatrix} x^{i_1}_{j_1} & \cdot & x^{i_1}_{s_\sigma} & \cdot & x^{i_1}_{j_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^{r_\rho}_{j_1} & \cdot & 0 & \cdot & x^{r_\rho}_{j_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^{i_k}_{j_1} & \cdot & x^{i_k}_{s_\sigma} & \cdot & x^{i_k}_{j_k} \end{vmatrix} .$$

Multiplying the u-th row, which consists of $x^{r_\rho}_{j_1}, \ldots, x^{r_\rho}_{j_{v-1}}$, 0, $x^{r_\rho}_{j_{v+1}}$, $\ldots$, $x^{r_\rho}_{j_k}$, and the v-th column of the last determinant above by X^i_j and employing (7.3.2) and (7.3.3), we obtain after some simplification,

$$(X^i_j)^2 X^{ur_\rho}_{vs_\sigma} = (X^i_j)^2 x^{r_\rho}_{s_\sigma} \tilde{x}^{i_u}_{j_v} + \sum_{\xi,\eta=1}^{k} X^{\xi r_\rho}_j X^i_{\eta s_\sigma} \begin{vmatrix} x^{i_1}_{j_1} & \cdot & x^{i_1}_{j_\eta} & \cdot & x^{i_1}_{j_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^{i_\xi}_{j_1} & \cdot & 0 & \cdot & x^{i_\xi}_{j_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^{i_k}_{j_1} & \cdot & x^{i_k}_{j_\eta} & \cdot & x^{i_k}_{j_k} \end{vmatrix}$$

$$= (X^i_j)^2 x^{r_\rho}_{s_\sigma} \tilde{x}^{i_u}_{j_v} + \sum_{\xi,\eta=1}^{k} X^{\xi r_\rho}_j X^i_{\eta s_\sigma} (X^{ui_\xi}_{vj_\eta} - x^{i_\xi}_{j_\eta} \tilde{x}^{i_u}_{j_v}).$$

Now $X^{ui_\xi}_{vj_\eta}$ vanishes unless $\xi = u$ and $\eta = v$, in which case it has the value X^i_j. Therefore we have

$$(X^i_j)^2 X^{ur_\rho}_{vs_\sigma} = (X^i_j)^2 x^{r_\rho}_{s_\sigma} \tilde{x}^{i_u}_{j_v} + X^i_j X^{ur_\rho}_j X^i_{vs_\sigma} - \sum_{\xi,\eta=1}^{k} X^{\xi r_\rho}_j X^i_{\eta s_\sigma} x^{i_\xi}_{j_\eta} \tilde{x}^{i_u}_{j_v} .$$

Multiplying through by $x^{i_u}_{j_v}$ and summing over u (or v), we obtain then, after rearranging terms,

(7.3.8) $$(X^i_j)^3 x^{r_\rho}_{s_\sigma}$$

$$= X^i_j \left\{ \sum_{u (\text{or } v)=1}^{k} (X^i_j X^{ur_\rho}_{vs_\sigma} - X^{ur_\rho}_j X^i_{vs_\sigma}) x^{i_u}_{j_v} + \sum_{\xi,\eta=1}^{k} X^{\xi r_\rho}_j x^{i_\xi}_{j_\eta} X^i_{\eta s_\sigma} \right\} .$$

Now if we next sum over v (or u), we obtain the more symmetric, if less useful, formula

$$(7.3.9) \quad k(x^i_j)^3 \, x^{r_\rho}_{s_\sigma} = x^i_j \sum_{u,v=1}^{k} (x^i_j \, X^{ur_\rho}_{vs_\sigma} + (k-1) \, X^{ur_\rho}_j \, X^i_{vs_\sigma}) \, x^{i_u}_{j_v}.$$

It is not difficult to verify the fact that (7.3.9) includes (7.3.2) and (7.3.3) as special cases.

Let X be an arbitrary $m \times n$ matrix of rank r, where m, n, and r are all greater than k. Since $r > k$, there is at least one non-singular submatrix of order $k+1$ in X; let this submatrix be that common to the rows $a_1, \ldots, a_{k+1}$ and columns $b_1, \ldots, b_{k+1}$. Since $X^{a_1 \ldots a_{k+1}}_{b_1 \ldots b_{k+1}} \neq 0$, it is clear that for some a and b, say a_α and b_β, it must be true that

$$x^{a_\alpha}_{b_\beta} \, X^{a_1 \ldots a_{\alpha-1} \; a_{\alpha+1} \ldots a_{k+1}}_{b_1 \ldots b_{\beta-1} \; b_{\beta+1} \ldots b_{k+1}} \neq 0 \; .$$

If we set $i_0 = a_\alpha$, $j_0 = b_\beta$, $i_u = a_u$ for $0 < u < \alpha$, $j_v = b_v$ for $0 < v < \beta$, $i_u = a_{u+1}$ for $\alpha < u < k+1$, and $j_v = b_{v+1}$ for $\beta < v < k+1$, we have

$$(7.3.10) \qquad x^{i_0}_{j_0} \, X^{i_1 \ldots i_k}_{j_1 \ldots j_k} \, X^{i_1 \ldots i_{\alpha-1} \; i_0 \; i_\alpha \ldots i_k}_{j_1 \ldots j_{\beta-1} \; j_0 \; j_\beta \ldots j_k} \neq 0 \; .$$

And therefore, since $X^{i_1 \ldots i_k}_{j_1 \ldots j_k} = X^i_j$ is different from zero, it follows from (7.3.9) that, for every element x^r_s of X,

$$(7.3.11) \quad x^r_s = \sum_{u,v=1}^{k} \left(\frac{1}{k} \, \frac{X^{ur}_{vs}}{X^i_j} + \frac{k-1}{k} \, \frac{X^{ur}_j}{X^i_j} \, \frac{X^i_{vs}}{X^i_j} \right) x^{i_u}_{j_v} \qquad .$$

But, since $X_{b_1\cdots b_{k+1}}^{a_1\cdots a_{k+1}} \neq 0$, we know by the argument presented in the first paragraph of the present section, that

$$(7.3.12) \qquad x_{b_v}^{a_u} = \left[\left[a\left[[X]_{b_1\cdots b_{k+1}}^{a_1\cdots a_{k+1}}\right]^{(k)}\right]^{(k)}\right]_v^u$$

for $u,v = 1,\ldots,k+1$, where $a^k(X_{b_1\cdots b_{k+1}}^{a_1\cdots a_{k+1}})^{k-1} = 1$. In other words, $x_{b_v}^{a_u}$ is the element common to the u-th row and the v-th column of the k-th compound of the matrix which is a times the k-th compound of the submatrix common to the rows $a_1,\ldots,a_{k+1}$ and the columns $b_1,\ldots,b_{k+1}$ of X. Now, since the ordered sets $\{i_1,\ldots,i_{\alpha-1},i_0,i_\alpha,\ldots,i_k\}$ and $\{j_1,\ldots,j_{\beta-1},j_0,j_\beta,\ldots,j_k\}$ coincide respectively with the ordered sets $\{a_1,\ldots,a_{k+1}\}$ and $\{b_1,\ldots,b_{k+1}\}$, it is clear that, by using (7.3.12) in (7.3.11), one can express every element of X in terms of the elements of $X^{(k)}$.

In closing this section, we remark that the first matrix in the right member of (7.3.5) and the second in that of (7.3.4) are examples of compounds considered by Bazin in 1851, see [1, p. 101].

<u>7.4. Solution of $X^{(k)} = B$ When B has Rank Greater Than k.</u> The following theorem is the principal

theorem of this chapter.

(7.4.1) Theorem. A necessary and sufficient condition that any $C(m,k)\times C(n,k)$ matrix B, of rank greater than k, whose elements $B^{i_1\cdots i_k}_{j_1\cdots j_k}$ are complex numbers and are skew-symmetric in both their upper and lower suffixes, be the k-th compound of some $m\times n$ matrix over the field of complex numbers is that the following conditions be satisfied simultaneously:

(7.4.1.1) the row, and column, vectors of B be k-vectors, i.e.,

$$B^{i_1\cdots i_k}_{j_1\cdots j_k}B^{r_1\cdots r_k}_{j_1\cdots j_k}=\sum_{u=1}^{k}B^{i_1\cdots i_{u-1}r_\rho i_{u+1}\cdots i_k}_{j_1\cdots j_k}B^{r_1\cdots r_{\rho-1}i_u r_{\rho+1}\cdots r_k}_{j_1\cdots j_k}$$

and

$$B^{i_1\cdots i_k}_{j_1\cdots j_k}B^{i_1\cdots i_k}_{s_1\cdots s_k}=\sum_{v=1}^{k}B^{i_1\cdots i_k}_{j_1\cdots j_{v-1}s_\sigma j_{v+1}\cdots j_k}B^{i_1\cdots i_k}_{s_1\cdots s_{\sigma-1}j_v s_{\sigma+1}\cdots s_k};$$

(7.4.1.2) in the abbreviated notation, where B^i_j, B^i_{vs}, B^{ur}_j, and B^{ur}_{vs} denote respectively $B^{i_1\cdots i_k}_{j_1\cdots j_k}$,

$B^{i_1\cdots i_k}_{j_1\cdots j_{v-1}sj_{v+1}\cdots j_k}$, $B^{i_1\cdots i_{u-1}ri_{u+1}\cdots i_k}_{j_1\cdots j_k}$, and

$B^{i_1\cdots i_{u-1}ri_{u+1}\cdots i_k}_{j_1\cdots j_{v-1}sj_{v+1}\cdots j_k}$, it be true that

$$B^i_j(B^i_jB^{u_1r_1}_{v_1s_1}-B^{u_1r_1}_jB^i_{v_1s_1})(B^i_jB^{u_2r_2}_{v_2s_2}-B^{u_2r_2}_jB^i_{v_2s_2})$$

$$=B^i_j(B^i_jB^{u_2r_1}_{v2s1}-B^{u_2r_1}_jB^i_{v2s1})(B^i_jB^{u_1r_2}_{v1s2}-B^{u_1r_2}_jB^i_{v1s2});$$

(7.4.1.3) if the sets $\{\alpha_1,\ldots,\alpha_{k-h-\lambda},\alpha'_1,\ldots,\alpha'_\lambda\}$, $\{\beta_1,\ldots,\beta_{k-h-\lambda},\beta'_1,\ldots,\beta'_\lambda\}$, $\{\xi_1,\ldots,\xi_\lambda,\ \xi'_1,\ldots,\xi'_{k-h-\lambda}\}$, and $\{\eta_1,\ldots,\eta_\lambda,\eta'_1,\ldots,\eta'_{k-h-\lambda}\}$ each consist of the first $k-h$ natural numbers in some order, then

$$\sum^\lambda (-1)^{p_\lambda}\Big\{ B^{i_{u_{\alpha_1}}\cdots i_{u_{\alpha_{k-h-\lambda}}} r_{\rho_{\xi_1}}\cdots r_{\rho_{\xi_\lambda}} i_{u'_1}\cdots i_{u'_k}}_{j_{v_1}\cdots j_{v_{k-h}} j_{v'_1}\cdots j_{v'_k}} B^{i_{u_{\alpha'_1}}\cdots i_{u_{\alpha'_\lambda}} r_{\rho_{\xi'_1}}\cdots r_{\rho_{\xi'_{k-h-\lambda}}} i_{u'_1}\cdots i_{u'_k}}_{s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_k}}$$

$$+ B^{i_{u_{\alpha_1}}\cdots i_{u_{\alpha_{k-h-\lambda}}} r_{\rho_{\xi_1}}\cdots r_{\rho_{\xi_\lambda}} i_{u'_1}\cdots i_{u'_k}}_{j_{v_1}\cdots j_{v_{\beta-1}} s_{\sigma_\eta} j_{v_{\beta+1}}\cdots j_{v_{k-h}} j_{v'_1}\cdots j_{v'_k}} B^{i_{u_{\alpha'_1}}\cdots i_{u_{\alpha'_\lambda}} r_{\rho_{\xi'_1}}\cdots r_{\rho_{\xi'_{k-h-\lambda}}} i_{u'_1}\cdots i_{u'_k}}_{s_{\sigma_1}\cdots s_{\sigma_{\eta-1}} j_{v_\beta} s_{\sigma_{\eta+1}}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_k}} \Big\} = 0$$

and

$$\sum_\lambda (-1)^{q_\lambda}\Big\{ B^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_k}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_k}} B^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_k}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_k}}$$

$$+ B^{i_{u_1}\cdots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_k}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_k}} B^{r_{\rho_1}\cdots r_{\rho_{\xi-1}} i_{u_\alpha} r_{\rho_{\xi+1}}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_k}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_k}} \Big\} = 0,$$

where $p_\lambda = \frac{1}{2}k(k+1) + k\lambda + (\alpha_1+\cdots+\alpha_{k-h-\lambda}) + (\xi_1+\cdots+\xi_\lambda)$,

$q = \frac{1}{2}k(k+1) + k\lambda + (\beta_1+\cdots+\beta_{k-h-\lambda}) + (\eta_1+\cdots+\eta_\lambda)$, $0 \leq h < k$,

$\sum^\lambda$ extends over all combinations of the integers $i_{u_1},\ldots,i_{u_{k-h}},r_{\rho_1},\ldots,r_{\rho_{k-h}}$ taken $k-h$ at a time, and

$\sum_\lambda$ extends over all combinations of the integers $j_{v_1},\ldots,j_{v_{k-h}},s_{\sigma_1},\ldots,s_{\sigma_{k-h}}$ taken $k-h$ at a time; and, finally, (7.4.1.4) there be a non-singular, "distinguished" submatrix of B of order k+1, where the term "distinguished" is to mean that the upper suffixes (and the lower suffixes) of its elements are the k-tuples, under lexicographical ordering, of an ordered set of k+1 distinct integers appearing among the upper suffixes (the lower suffixes) of the elements of B.

We shall prove the conditions of (7.4.1) necessary by showing that they hold for the k-th compound $X^{(k)}$ of any mxn matrix X of rank greater than k.

To prove the condition (7.4.1.1) necessary we could repeat the argument, with obvious modifications, used to prove (5.1.2). However, we can make a much shorter proof than that. To this end, let us consider the 2k-rowed determinant

$$\begin{vmatrix} x^{i_1}_{j_1} & \cdot & x^{i_1}_{j_k} & x^{i_1}_{s_\sigma} & 0 & \cdot & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x^{i_k}_{j_1} & \cdot & x^{i_k}_{j_k} & x^{i_k}_{s_\sigma} & 0 & \cdot & 0 & 0 & \cdot & 0 \\ x^{i_1}_{j_1} & \cdot & x^{i_1}_{j_k} & x^{i_1}_{s_\sigma} & x^{i_1}_{s_1} & \cdot & x^{i_1}_{s_{\sigma-1}} & x^{i_1}_{s_{\sigma+1}} & \cdot & x^{i_1}_{s_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x^{i_k}_{j_1} & \cdot & x^{i_k}_{j_k} & x^{i_k}_{s_\sigma} & x^{i_k}_{s_1} & \cdot & x^{i_k}_{s_{\sigma-1}} & x^{i_k}_{s_{\sigma+1}} & \cdot & x^{i_k}_{s_k} \end{vmatrix} .$$

If we expand it by Laplace's method, using the first k+1 columns, we see that it vanishes since every minor of order k+1, in the first k+1 columns has at least one row duplicated. On the other hand, if we expand it by Laplace's method, using the first k rows, we get

$$X^{i_1\cdots i_k}_{j_1\cdots j_k} X^{i_1\cdots i_k}_{s_\sigma s_1\cdots s_{\sigma-1}s_{\sigma+1}\cdots s_k}$$

$$+ \sum_{\nu=1}^{k} (-)^{q_v} X^{i_1\cdots i_k}_{j_1\cdots j_{v-1}j_{v+1}\cdots j_k s_\sigma} X^{i_1\cdots i_k}_{j_v s_1\cdots s_{\sigma-1}s_{\sigma+1}\cdots s_k},$$

where $q_v = k(k+1) + k+1-v$. It follows that

$$X^{i_1\cdots i_k}_{j_1\cdots j_k} X^{i_1\cdots i_k}_{s_\sigma s_1\cdots s_{\sigma-1}s_{\sigma+1}\cdots s_k}$$

$$= \sum_{\nu=1}^{k} (-)^{k-v} X^{i_1\cdots i_k}_{j_1\cdots j_{v-1}j_{v+1}\cdots j_k s_\sigma} X^{i_1\cdots i_k}_{j_v s_1\cdots s_{\sigma-1}s_{\sigma+1}\cdots s_k}$$

since $k(k+1)$ is an even integer. Furthermore, since, by the properties of determinants, the elements $X^{i_1\cdots i_k}_{j_1\cdots j_k}$ are clearly skew-symmetric in both their upper and lower suffixes, one obtains, by permuting columns,

$$X^{i_1\cdots i_k}_{j_1\cdots j_k} X^{i_1\cdots i_k}_{s_1\cdots s_k}$$

$$= \sum_{\nu=1}^{k} X^{i_1\cdots i_k}_{j_1\cdots j_{v-1}s_\sigma j_{v+1}\cdots j_k} X^{i_1\cdots i_k}_{s_1\cdots s_{\sigma-1}j_v s_{\sigma+1}\cdots s_k}.$$

And, using rows vice columns, one can show by an entirely similar argument that

$$X^{i_1\cdots i_k}_{j_1\cdots j_k} X^{r_1\cdots r_k}_{j_1\cdots j_k}$$

$$= \sum_{u=1}^{k} X^{i_1\cdots i_{u-1} r_\rho i_{u+1}\cdots i_k}_{j_1\cdots j_k} X^{r_1\cdots r_{\rho-1} i_u r_{\rho+1}\cdots r_k}_{j_1\cdots j_k} .$$

This proves the condition (7.4.1.1) necessary.

It follows directly from the equation immediately preceding (7.3.8) that

$$(7.4.2)\quad X^i_j \left(X^i_j X^{ur_\rho}_{vs_\sigma} - X^{ur_\rho}_j X^i_{vs_\sigma}\right)$$

$$= \left((X^i_j)^2 x^{r_\rho}_{s_\sigma} - \sum_{\xi,\eta=1}^{k} X^{\xi r_\rho}_j x^{i_\xi}_{j_\eta} X^i_{\eta s_\sigma}\right) x^{i_u}_{j_v} .$$

From this equation we derive two equations (one of them parenthetically) as follows: we replace u and v respectively by u_1 and v_1 (by u_2 and v_2) and then multiply both members by $\tilde{x}^{i_{u_2}}_{j_{v_2}}$ (by $\tilde{x}^{i_{u_1}}_{j_{v_1}}$). Since the derived equations have identical right members, their left members are equal, that is,

$$X^i_j \left(X^i_j X^{u_1 r_\rho}_{v_1 s_\sigma} - X^{u_1 r_\rho}_j X^i_{v_1 s_\sigma}\right) \tilde{x}^{i_{u_2}}_{j_{v_2}}$$

$$= X^i_j \left(X^i_j X^{u_2 r_\rho}_{v_2 s_\sigma} - X^{u_2 r_\rho}_j X^i_{v_2 s_\sigma}\right) \tilde{x}^{i_{u_1}}_{j_{v_1}} .$$

If, in this equation, one first takes $\rho = 1$ and $\sigma = 1$ then takes $\rho = 2$ and $\sigma = 2$, he obtains, upon cross multiplying the resulting equations,

$$(X^{i}_{j})^{2}(X^{i}_{j}X^{u_1r_1}_{v_1s_1} - X^{u_1r_1}_{j}X^{i}_{v_1s_1})(X^{i}_{j}X^{u_2r_2}_{v_2s_2} - X^{u_2r_2}_{j}X^{i}_{v_2s_2})\tilde{x}^{iu_1}_{jv_1}\tilde{x}^{iu_2}_{jv_2}$$

$$= (X^{i}_{j})^{2}(X^{i}_{j}X^{u_2r_1}_{v_2s_1} - X^{u_2r_1}_{j}X^{i}_{v_2s_1})(X^{i}_{j}X^{u_1r_2}_{v_1s_2} - X^{u_1r_2}_{j}X^{i}_{v_1r_2})\tilde{x}^{iu_1}_{jv_1}\tilde{x}^{iu_2}_{jv_2}.$$

Therefore it is true that

$$(7.4.3)\quad X^{i}_{j}(X^{i}_{j}X^{u_1r_1}_{v_1s_1} - X^{u_1r_1}_{j}X^{i}_{v_1s_1})(X^{i}_{j}X^{u_2r_2}_{v_2s_2} - X^{u_2r_2}_{j}X^{i}_{v_2s_2})$$

$$= X^{i}_{j}(X^{i}_{j}X^{u_1r_2}_{v_1s_2} - X^{u_1r_2}_{j}X^{i}_{v_1s_2})(X^{i}_{j}X^{u_2r_1}_{v_2s_1} - X^{u_2r_1}_{j}X^{i}_{v_2s_1}) .$$

For, if $X^{i}_{j}\,\tilde{x}^{iu_1}_{jv_1}\,\tilde{x}^{iu_2}_{jv_2} \neq 0$, (7.4.3) is obtained by dividing both members of the preceding equation by $X^{i}_{j}\,\tilde{x}^{iu_1}_{jv_1}\,\tilde{x}^{iu_2}_{jv_2}$. Otherwise, both members of (7.4.3) vanish in one or more of the following ways: when $X^{i}_{j} = 0$, the first factor in each is zero and when $\tilde{x}^{iu_1}_{jv_1} = 0$ (or $\tilde{x}^{iu_2}_{jv_2} = 0$), the second factor (or the third factor) in each member is, by (7.4.2), zero. This proves the condition (7.4.1.2) necessary.

To show that the second relation of (7.4.1.3) is necessary, we consider the 2k-rowed determinant

$$
\begin{vmatrix}
x^{i_{u_1}}_{j_{v_1}} & \cdot & x^{i_{u_1}}_{j_{v_{k-h}}} & x^{i_{u_1}}_{s_{\sigma_1}} & \cdot & x^{i_{u_1}}_{s_{\sigma_{k-h}}} & x^{i_{u_1}}_{j_{v'_1}} & \cdot & x^{i_{u_1}}_{j_{v'_h}} & 0 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^{i_{u_{k-h}}}_{j_{v_1}} & \cdot & x^{i_{u_{k-h}}}_{j_{v_{k-h}}} & x^{i_{u_{k-h}}}_{s_{\sigma_1}} & \cdot & x^{i_{u_{k-h}}}_{s_{\sigma_{k-h}}} & x^{i_{u_{k-h}}}_{j_{v'_1}} & \cdot & x^{i_{u_{k-h}}}_{j_{v'_h}} & 0 & \cdot & 0 \\
x^{i_{u'_1}}_{j_{v'_1}} & \cdot & x^{i_{u'_1}}_{j_{v_{k-h}}} & x^{i_{u'_1}}_{s_{\sigma_1}} & \cdot & x^{i_{u'_1}}_{s_{\sigma_{k-h}}} & x^{i_{u'_1}}_{j_{v'_1}} & \cdot & x^{i_{u'_1}}_{j_{v'_h}} & 0 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^{i_{u'_h}}_{j_{v_1}} & \cdot & x^{i_{u'_h}}_{j_{v_{k-h}}} & x^{i_{u'_h}}_{s_{\sigma_1}} & \cdot & x^{i_{u'_h}}_{s_{\sigma_{k-h}}} & x^{i_{u'_h}}_{j_{v'_1}} & \cdot & x^{i_{u'_h}}_{j_{v'_h}} & 0 & \cdot & 0 \\
x^{r_{\rho_1}}_{j_{v_1}} & \cdot & x^{r_{\rho_1}}_{j_{v_{k-h}}} & x^{r_{\rho_1}}_{s_{\sigma_1}} & \cdot & x^{r_{\rho_1}}_{s_{\sigma_{k-h}}} & 0 & \cdot & 0 & x^{r_{\rho_1}}_{j_{v'_1}} & \cdot & x^{r_{\rho_1}}_{j_{v'_h}} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^{r_{\rho_{k-h}}}_{j_{v_1}} & \cdot & x^{r_{\rho_{k-h}}}_{j_{v_{k-h}}} & x^{r_{\rho_{k-h}}}_{s_{\sigma_1}} & \cdot & x^{r_{\rho_{k-h}}}_{s_{\sigma_{k-h}}} & 0 & \cdot & 0 & x^{r_{\rho_{k-h}}}_{j_{v'_1}} & \cdot & x^{r_{\rho_{k-h}}}_{j_{v'_h}} \\
x^{i_{u'_1}}_{j_{v_1}} & \cdot & x^{i_{u'_1}}_{j_{v_{k-h}}} & x^{i_{u'_1}}_{s_{\sigma_1}} & \cdot & x^{i_{u'_1}}_{s_{\sigma_{k-h}}} & 0 & \cdot & 0 & x^{i_{u'_1}}_{j_{v'_1}} & \cdot & x^{i_{u'_1}}_{j_{v'_h}} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x^{i_{u'_h}}_{j_{v_1}} & \cdot & x^{i_{u'_h}}_{j_{v_{k-h}}} & x^{i_{u'_h}}_{s_{\sigma_1}} & \cdot & x^{i_{u'_h}}_{s_{\sigma_{k-h}}} & 0 & \cdot & 0 & x^{i_{u'_h}}_{j_{v'_1}} & \cdot & x^{i_{u'_h}}_{j_{v'_h}}
\end{vmatrix}
$$

which we shall call the "special" determinant. From this determinant we derive the following special relation.

$$(7.4.4)\quad X^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h} r_{\rho_1}\cdots r_{\rho_{k-h}}}_{j_{v_1}\cdots j_{v_{k-h}} s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_h}}\; X^{i_{u'_1}\cdots i_{u'_h}}_{j_{v'_1}\cdots j_{v'_h}} = \Sigma_\lambda (-)^{q_\lambda} X^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_h}}\; X^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_h}}.$$

To obtain the right member of (7.4.4), expand the "special" determinant by Laplace's method, using the first k rows. Clearly, $q_\lambda = \frac{1}{2}k(k+1) + k\lambda + (\beta_1+\dots+\beta_{k-h-\lambda}) + (\eta_1+\dots+\eta_\lambda)$ and Σ_λ extends over all combinations of the integers

$$j_{v_1},\dots,j_{v_{k-h}},s_{\sigma_1},\dots,s_{\sigma_{k-h}}$$

taken $k-h$ at a time. If $h = 0$, the second factor of the left member is to be taken as unity and the first factor is then easily seen to be the "special" determinant itself. Otherwise, we obtain the left member of (7.4.4) by first augmenting each of the elements of the $(2k-2h+t)$-th column of the "special" determinant by the corresponding element of its $(2k-h+t)$-th column, for $t = 1,2,\dots,h$, and then expanding the resulting determinant by Laplace's method, using the first $2k-h$ columns of which every minor of order $2k-h$ except the first has a row duplicated unless its algebraic complement has a row of zeros. Now, if, in the "special" determinant, we replace $x_t^{i_{u_\alpha}}$ by $x_t^{r_{\rho_\xi}}$ and vice versa, for

$$t = j_{v_1}, \ldots, j_{v_{k-h}}, s_{\sigma_1}, \ldots, s_{\sigma_{k-h}}, j_{v'_1}, \ldots, j_{v'_h} \quad ,$$

we obtain a new determinant which we can evaluate in the same two ways in which we evaluated the "special" determinant. We obtain, as a result,

$$\chi^{i_{u_1}\cdots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h} r_{\rho_1}\cdots r_{\rho_{\xi-1}} i_{u_\alpha} r_{\rho_{\xi+1}}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_1}\cdots j_{v_{k-h}} s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_h}} \chi^{i_{u'_1}\cdots i_{u'_h}}_{j_{v'_1}\cdots j_{v'_h}}$$

$$= \sum_\lambda (-)^{q_\lambda} \chi^{i_{u_1}\cdots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_h}} \chi^{r_{\rho_1}\cdots r_{\rho_{\xi-1}} i_{u_\alpha} r_{\rho_{\xi+1}}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_h}} .$$

Comparing this equation with (7.4.4), we see that their left members differ only in sign due to an interchange of a single pair of rows in the first factor and, therefore, the sum of their right members must be zero. That is,

$$\sum_\lambda (-)^{q_\lambda} \Big\{ \chi^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_h}} \chi^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_h}} +$$

$$\chi^{i_{u_1}\cdots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_h}} \chi^{r_{\rho_1}\cdots r_{\rho_{\xi-1}} i_{u_\alpha} r_{\rho_{\xi+1}}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_h}} \Big\} = 0 .$$

This proves the second relation of condition (7.4.1.3) necessary. By an entirely similar argument, one can show that the first relation of (7.4.1.3) is also necessary.

Finally, the condition (7.4.1.4) is necessary because, by (1.4.4), any matrix X such that $X^{(k)} = B$ must have rank greater than k when B has rank greater than k, and, therefore, there must exist some non-singular submatrix of X having order $k+1$, say $[X]^{a_1 \dots a_{k+1}}_{b_1 \dots b_{k+1}}$, whose k-th compound is then, by (1.4.4) and the meaning ascribed to the term "distinguished" in (7.4.1.4), a non-singular, "distinguished" submatrix of $X^{(k)}$.

This completes the proof of the fact that the conditions of (7.4.1) are necessary conditions. We shall prove that they constitute a sufficient set of conditions by showing how to construct an $m \times n$ matrix X whose k-th compound equals the given matrix B.

By (7.4.1.4), there is a non-singular, "distinguished" submatrix of B of order $k+1$, say

$$\begin{bmatrix} B^{a_1 \dots a_k}_{b_1 \dots b_k} & \cdot & B^{a_1 \dots a_k}_{b_2 \dots b_{k+1}} \\ \cdot & \cdot & \cdot \\ B^{a_2 \dots a_{k+1}}_{b_1 \dots b_k} & \cdot & B^{a_2 \dots a_{k+1}}_{b_2 \dots b_{k+1}} \end{bmatrix},$$

which, for brevity, we shall denote by A. Let a be a complex number such that $a^k / |A|^{k-1} = 1$, which is possible since $|A| \neq 0$, and set

$$(7.4.5)\quad C \equiv \begin{bmatrix} c^{a_1}_{b_1} & \cdot & c^{a_1}_{b_{k+1}} \\ \cdot & \cdot & \cdot \\ c^{a_{k+1}}_{b_{k+1}} & \cdot & c^{a_{k+1}}_{b_{k+1}} \end{bmatrix} = a\, A^{(k)} \quad .$$

Then it follows from (7.2.1) that

$$(7.4.6)\qquad C^{a_{u_1}\cdots a_{u_k}}_{b_{v_1}\cdots b_{v_k}} = B^{a_{u_1}\cdots a_{u_k}}_{b_{v_1}\cdots b_{v_k}}$$

for every pair $(u_1,\ldots,u_k)$, $(v_1,\ldots,v_k)$ of k-tuples of the first $k+1$ natural numbers. We need not insist on lexicographical ordering in (7.4.6) since the C's and B's are skew-symmetric in their suffixes.

Since A is non-singular, it follows from (7.4.5), in view of (1.4.4), that the expansion of $|C|$ by elements and cofactors of the α-th row must contain at least one non-vanishing term, say $c^{a_\alpha}_{b_\beta}\, \tilde{c}^{a_\alpha}_{b_\beta} \neq 0$, where $\tilde{c}^{a_\alpha}_{b_\beta}$ is the cofactor of $c^{a_\alpha}_{b_\beta}$ in $|C|$. For notational convenience, we put

$$(7.4.7)\quad i_1 = a_1,\ldots,i_{\alpha-1} = a_{\alpha-1}, i_0 = a_\alpha, i_\alpha = a_{\alpha+1},\ldots,i_k = a_{k+1};\ \ j_1 = b_1,\ldots,j_{\beta-1} = b_{\beta-1}, j_0 = b_\beta, j_\beta = b_{\beta+1},\ldots,j_k = b_{k+1};$$

and have, as a consequence,

$$(7.4.8)\qquad c^{i_0}_{j_0}\, C^{i_1\cdots i_k}_{j_1\cdots j_k} \neq 0 \ .$$

Furthermore, we see by (7.4.8), that the expansion of $C^{i_1 \cdots i_k}_{j_1 \cdots j_k}$ by elements and cofactors of the π-th row must have at least one non-vanishing term, say $c^{i_\pi}_{j_\gamma} \tilde{c}^{i_\pi}_{j_\gamma} \neq 0$, where $\tilde{c}^{i_\pi}_{j_\gamma}$ is the cofactor of $c^{i_\pi}_{j_\gamma}$ in $C^{i_1 \cdots i_k}_{j_1 \cdots j_k}$.

From this it follows that

$$(7.4.9) \qquad c^{i_\pi}_{j_\gamma} \, C^{i_1 \cdots i_{\pi-1} i_{\pi+1} \cdots i_k}_{j_1 \cdots j_{\gamma-1} j_{\gamma+1} \cdots j_k} \neq 0 \quad .$$

Since, by (7.4.6) and (7.4.7), $B^{i_1 \cdots i_k}_{j_1 \cdots j_k} = C^{i_1 \cdots i_k}_{j_1 \cdots j_k}$, it follows from (7.4.8) that $B^{i_1 \cdots i_k}_{j_1 \cdots j_k} \neq 0$. Then, using the abbreviated notation, the $m \times n$ matrix X which we set out to construct is that whose r,s-th element is defined by

$$(7.4.10) \qquad x^r_s = \sum_{u,v=1}^{k} \left(\frac{1}{k} \frac{B^{ur}_{vs}}{B^i_j} + \frac{k-1}{k} \frac{B^{ur}_j}{B^i_j} \frac{B^i_{vs}}{B^i_j} \right) c^{i_u}_{j_v} \quad .$$

Now, to complete the proof of (7.4.1), we must show that

$$(7.4.11) \qquad X^{r_1 \cdots r_k}_{s_1 \cdots s_k} = B^{r_1 \cdots r_k}_{s_1 \cdots s_k}$$

holds for every element $X^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ of the k-th compound of X. To this end, we establish several lemmas.

It follows from (7.4.6), as modified by (7.4.7), and (7.4.8) that the following lemma is true.

(7.4.12) Lemma. In the abbreviated notation, we have:

$$C^{i}_{j} = B^{i}_{j},\ C^{ui_0}_{vj_0} = B^{ui_0}_{vj_0},\ C^{ui_0}_{j} = B^{ui_0}_{j},\ \text{and}\ C^{i}_{vj_0} = B^{i}_{vj_0}$$

for $u,v = 1,\ldots,k$.

Let $\tilde{c}^{i_u}_{j_v}$ denote the cofactor of $c^{i_u}_{j_v}$ in $C^{i_1\ldots i_k}_{j_1\ldots j_k}$.

Then, by (7.4.5) and (7.4.7), we have

$$\tilde{c}^{i_u}_{j_v} = (-)^{u+v} \left| \left[aA^{(k)}\right]^{1\ldots\alpha_1-1,\alpha_1+1\ldots\alpha_2-1,\alpha_2+1\ldots k+1}_{1\ldots\beta_1-1,\beta_1+1\ldots\beta_2-1,\beta_2+1\ldots k+1} \right|,$$

where $\alpha_1 = \min(u,\alpha), \alpha_2 = \max(u,\alpha), \beta_1 = \min(v,\beta), \beta_2 = \max(v,\beta)$.

If we factor a from each of its rows and apply (2.2.2) to the determinant in the right member of the last equation above, we obtain, if we make use of the fact that $\alpha_1+\alpha_2 = u+\alpha$ and $\beta_1+\beta_2 = v+\beta$,

$$\tilde{c}^{i_u}_{j_v} = (-)^{\alpha+\beta}\, a^{k-1} |A|^{k-2} \left| \left[\operatorname{adj}^{(k)} A'\right]^{\alpha_1\alpha_2}_{\beta_1\beta_2} \right| .$$

If we multiply both members of this equation by $a|A|$, remembering that $a^k|A|^{k-1} = 1$, and make use of the fact that, in view of Section 1.3 and the definition of A,

$$\left[\operatorname{adj}^{(k)} A'\right]^{\alpha_\xi}_{\beta_\eta} = (-)^{\alpha_\xi+\beta_\eta} B^{a_1\ldots a_{\alpha_\xi-1}a_{\alpha_\xi+1}\ldots a_{k+1}}_{b_1\ldots b_{\beta_\eta-1}b_{\beta_\eta+1}\ldots b_{k+1}},$$

we obtain, by expanding the determinant in its right member,

$$a\,|A|\,\tilde{c}^{i_u}_{j_v} =$$

$$(-)^{u+v}\left(B^{a_1\cdots a_{\alpha_1-1}a_{\alpha_1+1}\cdots a_{k+1}}_{b_1\cdots b_{\beta_1-1}b_{\beta_1+1}\cdots b_{k+1}}\, B^{a_1\cdots a_{\alpha_2-1}a_{\alpha_2+1}\cdots a_{k+1}}_{b_1\cdots b_{\beta_2-1}b_{\beta_2+1}\cdots b_{k+1}} - B^{a_1\cdots a_{\alpha_2-1}a_{\alpha_2+1}\cdots a_{k+1}}_{b_1\cdots b_{\beta_1-1}b_{\beta_1+1}\cdots b_{k+1}}\, B^{a_1\cdots a_{\alpha_1-1}a_{\alpha_1+1}\cdots a_{k+1}}_{b_1\cdots b_{\beta_2-1}b_{\beta_2+1}\cdots b_{k+1}}\right).$$

Using (7.4.7) and changing, if necessary, the order of the factors of the terms in the parentheses, we obtain

$$a\,|A|\,\tilde{c}^{i_u}_{j_v} = (-)^{u+v}\left(B^{i_1\cdots i_k}_{j_1\cdots j_k}\, B^{i_1\cdots i_{u-1}i_{u+1}\cdots i_{\alpha-1}i_0 i_\alpha\cdots i_k}_{j_1\cdots j_{v-1}j_{v+1}\cdots j_{\beta-1}j_0 j_\beta\cdots j_k} - B^{i_1\cdots i_{u-1}i_{u+1}\cdots i_{\alpha-1}i_0 i_\alpha\cdots i_k}_{j_1\cdots j_k}\, B^{i_1\cdots i_k}_{j_1\cdots j_{v-1}j_{v+1}\cdots j_{\beta-1}j_0 j_\beta\cdots j_k}\right).$$

From this, by a judicious interchange of suffixes, we obtain, since the B's are skew-symmetric in their suffixes,

$$a\,|A|\,\tilde{c}^{i_u}_{j_v} = (-)^{\alpha+\beta}\left(B^{i_1\cdots i_k}_{j_1\cdots j_k}\, B^{i_1\cdots i_{u-1}i_0 i_{u+1}\cdots i_k}_{j_1\cdots j_{v-1}j_0 j_{v+1}\cdots j_k} - B^{i_1\cdots i_{u-1}i_0 i_{u+1}\cdots i_k}_{j_1\cdots j_k}\, B^{i_1\cdots i_k}_{j_1\cdots j_{v-1}j_0 j_{v+1}\cdots j_k}\right).$$

Therefore, since $a\,|A| \neq 0$, the next lemma is true.

(7.4.13) <u>Lemma.</u> If $\tilde{c}^{i_u}_{j_v}$ is the cofactor of $c^{i_u}_{j_v}$ in $C^{i_1\cdots i_k}_{j_1\cdots j_k}$, then, in the abbreviated notation,

$$\tilde{c}^{i_u}_{j_v} = (-)^{\alpha+\beta}\left(B^{i}_{j}B^{ui_0}_{vj_0} - B^{ui_0}_{j}\,B^{i}_{vj_0}\right) / (a\,|A|)\,.$$

It follows from (7.4.9) and (7.4.13) that the next lemma is true.

(7.4.14) Lemma. $B_j^i B_{\nu j_0}^{\pi i_0} - B_j^{\pi i_0} B_{\nu j_0}^i \neq 0.$

By the first relation of (7.4.1.1), we know that

$$(B_j^i)^k B_{j_1\cdots j_k}^{r_1\cdots r_k} = (B_j^i)^{k-1} \sum_{u_1=1}^{k} B_j^{u_1 r_1} B_j^{i_{u_1} r_2\cdots r_k}$$

$$= (B_j^i)^{k-2} \sum_{u_1=1}^{k} B_j^{u_1 r_1} \sum_{u_2=1}^{k} B_j^{u_2 r_2} B_j^{i_{u_1} i_{u_2} r_3\cdots r_k}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$= \sum_{u_1=1}^{k} B_j^{u_1 r_1} \sum_{u_2=1}^{k} B_j^{u_2 r_2} \cdots \sum_{u_k=1}^{k} B_j^{u_k r_k} B_j^{i_{u_1}\cdots i_{u_k}} .$$

Since the factor $B_j^{i_{u_1}\cdots i_{u_k}}$ vanishes unless the upper suffixes are all different and then has the value $+ B_j^i$ or $- B_j^i$ according as $(u_1,\ldots,u_k)$ is an even or an odd permutation of $(1,\ldots,k)$, it follows that

$$(B_j^i)^k B_{j_1\cdots j_k}^{r_1\cdots r_k} = B_j^i \sum \pm B_j^{u_1 r_1} B_j^{u_2 r_2} \cdots B_j^{u_k r_k} ,$$

where the summation extends over all permutations $(u_1,\ldots,u_k)$ of $(1,\ldots,k)$ and the sign is + or - according as the permutation is even or odd. An entirely similar argument based on the second relation in (7.4.1.1) leads to the equation

$$(B^i_j)^k \; B^{i_1 \cdots i_k}_{s_1 \cdots s_k} = B^i_j \sum \pm B^i_{v_1 s_1} \; B^i_{v_2 s_2} \cdots B^i_{v_k s_k} \quad .$$

Since the sums in the right members of these two equations can be written as determinants, we have as a consequence, the next lemma.

(7.4.15) <u>Lemma.</u> In the abbreviated notation,

$$(B^i_j)^k \; B^{r_1 \cdots r_k}_{j_1 \cdots j_k} = B^i_j \begin{vmatrix} B^{1r_1}_j & \cdot & B^{kr_1}_j \\ \cdot & \cdot & \cdot \\ B^{1r_k}_j & & B^{kr_k}_j \end{vmatrix}$$

and

$$(B^i_j)^k \; B^{i_1 \cdots i_k}_{s_1 \cdots s_k} = B^i_j \begin{vmatrix} B^i_{1s_1} & \cdot & B^i_{1s_k} \\ \cdot & \cdot & \cdot \\ B^i_{ks_1} & \cdot & B^i_{ks_k} \end{vmatrix} \quad .$$

Since the B's are skew-symmetric in their suffixes, we have, for $1 \leqq \xi \leqq k$, $B^{\xi i_\xi}_{vs} = B^i_{vs}$, $B^{\xi i_\xi}_j = B^i_j$, and, when $u \neq \xi$, $B^{ui_\xi}_{vs} = B^{ui_\xi}_j = 0$. Consequently, when $r = i_\xi$, $1 \leqq \xi \leqq k$, (7.4.10) reduces to

$$x^{i_\xi}_s = \sum_{\nu=1}^{k} (B^i_{vs} \,/\, B^i_j) \; c^{i_\xi}_{j_v} \quad .$$

Entirely similar arguments lead one to conclude, further, that

$$x_{j_\eta}^{r} = \sum_{u=1}^{k} (B_j^{ur} / B_j^{i})\, c_{j_\eta}^{i_u} \quad \text{and} \quad x_{j_\eta}^{i_\xi} = c_{j_\eta}^{i_\xi} \; .$$

Hence we have the next lemma.

(7.4.16) <u>Lemma.</u> In the abbreviated notation,

$$x_{j_\eta}^{r} = \sum_{u=1}^{k} (B_j^{ur} / B_j^{i})\, c_{j_\eta}^{i_u}, \; x_{s}^{i_\xi} = \sum_{v=1}^{k} (B_{vs}^{i} / B_j^{i})\, c_{j_v}^{i_\xi}, \text{ and}$$

$$x_{j_\eta}^{i_\xi} = c_{j_\eta}^{i_\xi} \qquad \text{for } \xi, \eta = 1, \ldots, k.$$

From the first relation in (7.4.16), we deduce

$$\begin{bmatrix} x_{j_1}^{r_1} & \cdot & x^{r_1} \\ \cdot & \cdot & \cdot \\ x_{j_1}^{r_k} & \cdot & x_{j_k}^{r_k} \end{bmatrix} = \begin{bmatrix} \sum_{u=1}^{k} (B_j^{ur_1} / B_j^{i}) c_{j_1}^{i_u} & \cdot & \sum_{u=1}^{k} (B_j^{ur_1} / B_j^{i}) c_{j_k}^{i_u} \\ \cdot & \cdot & \cdot \\ \sum_{u=1}^{k} (B_j^{ur_k} / B_j^{i}) c_{j_1}^{i_u} & \cdot & \sum_{u=1}^{k} (B_j^{ur_k} / B_j^{i}) c_{j_k}^{i_u} \end{bmatrix}$$

$$= \frac{1}{B_j^{i}} \begin{bmatrix} B_j^{1r_1} & \cdot & B_j^{kr_1} \\ \cdot & \cdot & \cdot \\ B_j^{1r_k} & \cdot & B_j^{kr_k} \end{bmatrix} \begin{bmatrix} c_{j_1}^{i_1} & \cdot & c_{j_k}^{i_1} \\ \cdot & \cdot & \cdot \\ c_{j_1}^{i_k} & \cdot & c_{j_k}^{i_k} \end{bmatrix} .$$

Multiplying through by B_j^{i} and taking determinants, we obtain

$$(B_j^i)^k X^{r_1\cdots r_k}_{j_1\cdots j_k} = C_j^i \begin{vmatrix} B_j^{1r_1} & \cdot & B_j^{kr_1} \\ \cdot & \cdot & \cdot \\ B_j^{1r_k} & \cdot & B_j^{kr_k} \end{vmatrix} .$$

Therefore it follows from the first relation of (7.4.15), since, by (7.4.12), $C_j^i = B_j^i$, that

$$(B_j^i)^k X^{r_1\cdots r_k}_{j_1\cdots j_k} = (B_j^i)^k B^{r_1\cdots r_k}_{j_1\cdots j_k} .$$

But we know, by (7.4.8) and (7.4.12), that $B_j^i \neq 0$. Hence

$$X^{r_1\cdots r_k}_{j_1\cdots j_k} = B^{r_1\cdots r_k}_{j_1\cdots j_k} .$$

Starting with the second relation in (7.4.16), we deduce by means of the second relation in (7.4.15), by an entirely similar method, a corresponding relation, namely,

$$X^{i_1\cdots i_k}_{s_1\cdots s_k} = B^{i_1\cdots i_k}_{s_1\cdots s_k} .$$

Since no restriction was put on $(r_1,\ldots,r_k)$ or $(s_1,\ldots,s_k)$, we have proved the next lemma.

(7.4.17) <u>Lemma.</u> For all k-tuples $(r_1,\ldots,r_k)$ and $(s_1,\ldots,s_k)$, it is true that

$$X^{r_1 \cdots r_k}_{j_1 \cdots j_k} = B^{r_1 \cdots r_k}_{j_1 \cdots j_k} \quad \text{and} \quad X^{i_1 \cdots i_k}_{s_1 \cdots s_k} = B^{i_1 \cdots i_k}_{s_1 \cdots s_k} ;$$

in particular, $X^{i}_{j} = B^{i}_{j} \neq 0$.

If we apply the argument leading up to (7.3.11) to the matrix C, we obtain, in view of (7.4.8),

$$\begin{aligned} c^{i_0}_{j_0} &= \sum_{u,v=1}^{k} \left(\frac{1}{k} \frac{C^{ui_0}_{vj_0}}{C^{i}_{j}} + \frac{k-1}{k} \frac{C^{ui_0}_{j}}{C^{i}_{j}} \frac{C^{i}_{vj_0}}{C^{i}_{j}} \right) c^{i_u}_{j_v} \\ &= \sum_{u,v=1}^{k} \left(\frac{1}{k} \frac{B^{ui_0}_{vj_0}}{B^{i}_{j}} + \frac{k-1}{k} \frac{B^{ui_0}_{j}}{B^{i}_{j}} \frac{B^{i}_{vj_0}}{B^{i}_{j}} \right) c^{i_u}_{j_v} && \text{by (7.4.12)} \\ &= x^{i_0}_{j_0} && \text{by (7.4.10)} . \end{aligned}$$

Then, using (7.3.3), (7.4.8), (7.4.12), and (7.4.16), we have

$$c^{i_0}_{j_\eta} = \sum_{u=1}^{k} (C^{ui_0}_{j} \;/\; C^{i}_{j})\; c^{i_u}_{j_\eta} = \sum_{u=1}^{k} (B^{ui_0}_{j} \;/\; B^{i}_{j})\; c^{i_u}_{j_\eta} = x^{i_0}_{j_\eta}, \quad \eta = 1, \ldots, k;$$

and, using (7.3.2), (7.4.8), (7.4.12), and (7.4.16), we have

$$c^{i_\xi}_{j_0} = \sum_{v=1}^{k} (C^{i}_{vj_0} \;/\; C^{i}_{j}) c^{i_\xi}_{j_v} = \sum_{v=1}^{k} (B^{i}_{vj_0} \;/\; B^{i}_{j}) c^{i_\xi}_{j_v} = x^{i_\xi}_{j_0}, \quad \xi = 1, \ldots, k.$$

It follows from these and the third relation of (7.4.16) that the next lemma is true.

(7.4.18) <u>Lemma.</u> The submatrix $[X]^{i_1 \cdots i_{\alpha-1} i_0 i_\alpha \cdots i_k}_{j_1 \cdots j_{\beta-1} j_0 j_\beta \cdots j_k}$ of X is equal to the matrix C.

It follows from (7.4.2) that

$$X^i_j(X^i_j X^{\pi r}_s - X^{\pi r}_j X^i_{\gamma s}) = \left((X^i_j)^2 x^r_s - \sum_{\xi,\eta=1}^{k} X^{\xi r}_j x^{i_\xi}_{j_\eta} X^i_{\eta s}\right) \tilde{x}^{i_\pi}_{j_\gamma} \quad .$$

Using (7.4.10), (7.4.17), and (7.4.18) to replace the X's by B's and the x's by c's wherever possible in this equation, we obtain after collecting like terms,

$$B^i_j(B^i_j X^{\pi r}_{\gamma s} - B^{\pi r}_j B^i_{\gamma s}) = \frac{1}{k} \tilde{c}^{i_\pi}_{j_\gamma} \sum_{u,v=1}^{k} (B^i_j B^{ur}_{vs} - B^{ur}_j B^i_{vs}) c^{i_u}_{j_v} \quad .$$

In this equation, we replace $(B^i_j B^{ur}_{vs} - B^{ur}_j B^i_{vs})$ by

$$(B^i_j B^{ui_0}_{vj_0} - B^{ui_0}_j B^i_{vj_0})(B^i_j B^{\pi r}_{\gamma s} - B^{\pi r}_j B^i_{\gamma s}) \,/\, (B^i_j B^{\pi i_0}_{\gamma j_0} - B^{\pi i_0}_j B^i_{\gamma j_0}),$$

which is possible in view of (7.4.1.2) since $B^i_j \neq 0$, and replace this, in turn, by

$$\tilde{c}^{i_u}_{j_v} (B^i_j B^{\pi r}_{\gamma s} - B^{\pi r}_j B^i_{\gamma s}) \,/\, \tilde{c}^{i_\pi}_{j_\gamma} \quad ,$$

because of (7.4.13), to obtain

$$B^i_j(B^i_j X^{\pi r}_{\gamma s} - B^{\pi r}_j B^i_{\gamma s}) = \frac{1}{k} \sum_{u,v=1}^{k} \tilde{c}^{i_u}_{j_v} (B^i_j B^{\pi r}_{\gamma s} - B^{\pi r}_j B^i_{\gamma s}) c^{i_u}_{j_v}$$

$$= C^{i}_{j}(B^{i}_{j}B^{\pi r}_{\tau s} - B^{\pi r}_{j}B^{i}_{\tau s}) \; .$$

Since, by (7.4.17), $B^{i}_{j} = C^{i}_{j} \neq 0$, it follows from this that the next lemma is true.

(7.4.19) <u>Lemma.</u> $X^{\pi r}_{\tau s} = B^{r}_{\tau s}$ for all r, s.

Using (7.4.3), (7.4.1.2), (7.4.17), (7.4.18), and (7.4.19), we have

$$X^{i}_{j}(X^{i}_{j}X^{ur}_{vs} - X^{ur}_{j}X^{i}_{vs})(X^{i}_{j}X^{\pi i_0}_{\tau j_0} - X^{\pi i_0}_{j}X^{i}_{\tau j_0})$$

$$= X^{i}_{j}(X^{i}_{j}X^{ui_0}_{vj_0} - X^{ui_0}_{j}X^{i}_{vj_0})(X^{i}_{j}X^{r}_{s} - X^{r}_{j}X^{i}_{s})$$

$$= B^{i}_{j}(B^{i}_{j}B^{ui_0}_{vj_0} - B^{ui_0}_{j}B^{i}_{vj_0})(B^{i}_{j}B^{\pi r}_{\tau s} - B^{\pi r}_{j}B^{i}_{\tau s})$$

$$= B^{i}_{j}(B^{i}_{j}B^{ur}_{vs} - B^{ur}_{j}B^{i}_{vs})(B^{i}_{j}B^{\pi i_0}_{\tau j_0} - B^{\pi i_0}_{j}B^{i}_{\tau j_0})$$

$$= X^{i}_{j}(X^{i}_{j}B^{ur}_{vs} - X^{ur}_{j}X^{i}_{vs})(X^{i}_{j}X^{\pi i_0}_{\tau j_0} - X^{\pi i_0}_{j}X^{i}_{\tau j_0}) \; ;$$

and therefore, since $X^{i}_{j} \neq 0$ by (7.4.17), we have $X^{ur}_{vs} = B^{ur}_{vs}$. The argument is valid for $u,v = 1,\ldots,k$, $r = 1,\ldots,m$, and $s = 1,\ldots,n$. Hence we have proved the next lemma.

(7.4.20) <u>Lemma.</u> $X^{ur}_{vs} = B^{ur}_{vs}$ for all u, v, r, s.

Making use of the fact that the X's and B's are skew-symmetric in their suffixes, we easily obtain a slightly more general lemma as a consequence of (7.4.17) and (7.4.20).

(7.4.21) <u>Lemma.</u> If, for an arbitrary element $X^{r_1 \dots r_k}_{s_1 \dots s_k}$ of the k-th compound of the $m \times n$ matrix X whose elements are defined by (7.4.10), g of the r's are i's, say $r_{\rho'_1} = i_{u'_1}, \dots, r_{\rho'_g} = i_{u'_g}$, and h of the s's are j's, say $s_{\sigma'_1} = j_{v'_1}, \dots, s_{\sigma'_h} = j_{v'_h}$, then $X^{r_1 \dots r_k}_{s_1 \dots s_k} = B^{r_1 \dots r_k}_{s_1 \dots s_k}$ provided $h = g = k-1$ or $0 \leqq h \leqq g = k$ or $0 \leqq g \leqq h = k$.

Now we can see by (7.4.21) that each of those elements of $X^{(k)}$ for which we must yet show that $X^{r_1 \dots r_k}_{s_1 \dots s_k} = B^{r_1 \dots r_k}_{s_1 \dots s_k}$ fall into one of three categories, namely: (1) $0 \leqq h = g < k-1$, (2) $0 \leqq h < g < k$, or (3) $0 \leqq g < h < k$. We shall consider these three categories separately.

Let $X^{r_1 \dots r_s}_{s_1 \dots s_k}$ be any element in the first category, $0 \leqq h = g < k-1$. Then, because it is skew-symmetric in its suffixes, we have

$$X^{r_1\cdots r_k}_{s_1\cdots s_k} = E^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_h}}\, X^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_h}} \tag{7.4.22}$$

where the first factor in the right member has the value $+1$ or -1 according as the k-tuples of its upper and lower suffixes are respectively permutations of $(r_1,\ldots,r_k)$ and $(s_1,\ldots,s_k)$ of like parity or of unlike parity. But, as shown earlier, the elements of the k-th compound of X satisfy the relations of (7.4.1.3). From the second of these we obtain, by transposing all but the term for which $\lambda = 0$ and then multiplying through by $(-1)^{q_0}$,

$$X^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_1}\cdots j_{v_{k-h}} j_{v'_1}\cdots j_{v'_h}}\, X^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_h}} = \sum_{\lambda=1}^{k-h}\sum_{\beta}\sum_{\eta}(-)^{q_\lambda+q_0+1}$$

$$\Big(X^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_h}}\, X^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_h}} +$$

$$X^{i_{u_1}\cdots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_h}}\, X^{r_{\rho_1}\cdots r_{\rho_{\xi-1}} i_{u_\alpha} r_{\rho_{\xi+1}}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_{k-h-}}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_h}} \Big)$$

$$- X^{i_{u_1}\cdots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_1}\cdots j_{v_{k-h}} j_{v'_1}\cdots j_{v'_h}}\, X^{r_{\rho_1}\cdots r_{\rho_{\xi-1}} i_{u_\alpha} r_{\rho_{\xi+1}}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_h}},$$

where $\sum_\beta$ and $\sum_\eta$ extend respectively over the $(k-h-\lambda)$-tuples $(\beta_1,\ldots,\beta_{k-h-\lambda})$ and the λ-tuples $(\eta_1,\ldots,\eta_\lambda)$ of the first $k-h$ natural numbers. Dividing both members of this equation by

$$X^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_1}\cdots j_{v_{k-h}} j_{v'_1}\cdots j_{v'_h}} = E^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_1}\cdots j_{v_{k-h}} j_{v'_1}\cdots j_{v'_h}} X^{i_1\cdots i_k}_{j_1\cdots j_k}$$

which by (7.4.17) is different from zero, we obtain an evaluation for the second factor of the left member. Using the result in (7.4.22), we obtain

$$(7.4.23) \quad X^{r_1\cdots r_k}_{s_1\cdots s_k} = (E_q \,/\, X^i_j)\Big(\sum_{\lambda=1}^{k-h}\sum_\beta\sum_\eta(-)^{q_\lambda+q_0+1}$$

$$\Big\{X^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_h}} X^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_h}}$$

$$+ X^{i_{u_1}\cdots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta_1}}\cdots j_{v_{\beta_{k-h-\lambda}}} s_{\sigma_{\eta_1}}\cdots s_{\sigma_{\eta_\lambda}} j_{v'_1}\cdots j_{v'_h}} X^{r_{\rho_1}\cdots r_{\rho_{\xi-1}} i_{u_\alpha} r_{\rho_{\xi+1}}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_{\beta'_1}}\cdots j_{v_{\beta'_\lambda}} s_{\sigma_{\eta'_1}}\cdots s_{\sigma_{\eta'_{k-h-\lambda}}} j_{v'_1}\cdots j_{v'_h}}\Big\}$$

$$- X^{i_{u_1}\cdots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_1}\cdots j_{v_{k-h}} j_{v'_1}\cdots j_{v'_h}} X^{r_{\rho_1}\cdots r_{\rho_{\xi-1}} i_{u_\alpha} r_{\rho_{\xi+1}}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_h}}\Big)$$

where $E_q = E^{i_{u_1}\cdots i_{u_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{j_{v_1}\cdots j_{v_{k-h}} j_{v'_1}\cdots j_{v'_h}} E^{r_{\rho_1}\cdots r_{\rho_{k-h}} i_{u'_1}\cdots i_{u'_h}}_{s_{\sigma_1}\cdots s_{\sigma_{k-h}} j_{v'_1}\cdots j_{v'_h}}$.

This proves the following lemma.

(7.4.24) <u>Lemma.</u> Let $X^{i_1\cdots i_k}_{j_1\cdots j_k}$ be any non-zero element of the k-th compound of X. Then by (7.4.23) any element whose upper and lower suffixes contain respectively at least h i's and at least h j's , $0 \leqq h < k-1$, can be evaluated in terms of elements each of which has either more than h i's in its upper and no less than h j's in its lower suffix or more than h j's in its lower and no less than h i's in its upper suffix.

Since there is nothing in the way in which it was derived to preclude more than h of the r's being i's, it is evident that (7.4.23) holds also for elements in the second category. Therefore let the left member of (7.4.23) be any element in the second category. Since the relation therein expressed holds for all ξ, α such that $0 < \xi \leqq k-h$ and $0 < \alpha \leqq k-h$, it holds, in particular, for any ξ such that $r_{\rho_\xi} = r_{u'_{h+t}}$, $0 < t \leqq g-h$, and any α such that $i_{u_\alpha} \neq i_{u'_{h+t}}$. Such ξ and α surely exist since $0 \leqq h < g < k$ for every element in the second category. But by (7.4.21) $r_{u'_{h+t}} = i_{u'_{h+t}}$. Consequently, every element in the right member of (7.4.23) whose upper suffix is $i_{u_1} \dots i_{u_{\alpha-1}} r_{\rho_\xi} i_{u_{\alpha+1}} \dots i_{k-h} i_{u'_1} \dots i_{u'_h}$ vanishes, since the integer $i_{u'_{h+t}}$ is repeated therein, because the X's are skew-symmetric in their upper suffixes. Therefore, for elements in the second category, (7.4.23) reduces to the simpler relation

$$(7.4.25)\quad X^{r_1 \cdots r_k}_{s_1 \cdots s_k} = (E'_q / X^i_j) \sum_{\lambda=1}^{k-h} \Sigma_\beta \Sigma_\eta (-)^{q_\lambda + q_0 + 1}$$

$$X^{i_{u''_1} \cdots i_{u''_{k-g}}\, i_{u'_1} \cdots i_{u'_g}}_{j_{v_{\beta_1}} \cdots j_{v_{\beta_{k-h-\lambda}}}\, s_{\sigma_{\eta_1}} \cdots s_{\sigma_{\eta_\lambda}}\, j_{v'_1} \cdots j_{v'_h}} \; X^{r_{\rho''_1} \cdots r_{\rho''_{k-g}}\, i_{u'_1} \cdots i_{u'_g}}_{j_{v_{\beta'_1}} \cdots j_{v_{\beta'_\lambda}}\, s_{\sigma_{\eta'_1}} \cdots s_{\sigma_{\eta'_{k-h-\lambda}}}\, j_{v'_1} \cdots j_{v'_h}},$$

where

$$E'_q = E_q\, \epsilon^{i_{u''_1} \cdots i_{u''_{k-g}}\, i_{u'_1} \cdots i_{u'_g}}_{i_{u_1} \cdots i_{u_{k-h}}\, i_{u'_1} \cdots i_{u'_h}}\; \epsilon^{r_{\rho''_1} \cdots r_{\rho''_{k-g}}\, i_{u'_1} \cdots i_{u'_g}}_{r_{\rho_1} \cdots r_{\rho_{k-g}}\, i_{u_1} \cdots i_{u_h}}.$$

This shows that the next lemma is true.

(7.4.26) <u>Lemma.</u> Let $X^{i_1 \cdots i_k}_{j_1 \cdots j_k}$ be any non-zero element of the k-th compound of X. Then by (7.4.25) any element whose upper and lower suffixes contain respectively at least g i's and at least h j's, $0 \leq h < g < k$, can be evaluated in terms of elements each of which has more than h j's in its lower and no less than g i's in its upper suffix.

Let us modify in the following manner the argument which was initiated in the second paragraph following (7.4.21) and terminated in (7.4.25). Replace h by g in (7.4.22) and apply this in turn to the first of the relations of (7.4.1.3), with h replaced by g, instead of the second, after multiplying through by $(-1)^{p_0}$ instead of $(-1)^{q_0}$ and summing $\sum_{\lambda=1}^{k-g} \Sigma_\alpha \Sigma_\xi$ instead of $\sum_{\lambda=1}^{k-h} \Sigma_\beta \Sigma_\eta$, with the upper suffixes treated as were the lower suffixes and vice versa, to obtain a relation entirely analogous to (7.4.23), with E_q replaced by E_p. We find the analogue of (7.4.24) to be precisely (7.4.24) with h replaced by g. Continuing the analogy, let the left member of the analogue of (7.4.23) be any element in the third category (instead of the second category). In the ensuing argument, we use η, β instead of ξ, α; $h-g$ instead of $g-h$; $s_{\sigma_\eta} = s_{v'_{g+t}}$ instead of $r_{\rho_\xi} = r_{u'_{h+t}}$; $j_{v_\beta} \neq j_{v'_{g+t}}$ instead of $i_{u_\alpha} \neq i_{u'_{h+t}}$; $0 = g \quad h \quad k$ instead of $0 \leqq h < g < k$; "third" instead of "second"; $s_{v'_{g+t}} = j_{v'_{g+t}}$ instead of $r_{u'_{h+t}} = i_{u'_{h+t}}$; and the analogue of (7.4.23) instead of (7.4.23). Thus we obtain the analogue of (7.4.25), namely,

$$(7.4.27) \quad X^{r_1 \cdots r_k}_{s_1 \cdots s_k} = (E'_p \,/\, X^i_j) \sum_{\lambda=1}^{k-g} \sum_\alpha \sum_\xi (-)^{p_\lambda + p_0 + 1}$$

$$X^{i_{u_{\alpha_1}} \cdots i_{u_{\alpha_{k-g-\lambda}}} r_{\rho_{\xi_1}} \cdots r_{\rho_{\xi_\lambda}} i_{u'_1} \cdots i_{u'_g}}_{j_{v''_1} \cdots j_{v''_{k-h}} \; j_{v'_1} \cdots j_{v'_h}} \; X^{i_{u_{\alpha'_1}} \cdots i_{u_{\alpha'_\lambda}} r_{\rho_{\xi'_1}} \cdots r_{\rho_{\xi'_{k-g-\lambda}}} i_{u'_1} \cdots i_{u'_g}}_{s_{\sigma''_1} \cdots s_{\sigma''_{k-h}} j_{v'_1} \cdots j_{v'_h}} .$$

This proves the analogue of (7.4.26) which follows.

(7.4.28) <u>Lemma.</u> Let $X^{i_1 \cdots i_k}_{j_1 \cdots j_k}$ be any non-zero element of the k-th compound of X. Then by (7.4.27) any element whose upper and lower suffixes contain respectively at least g i's and at least h j's, $0 \leqq g < h < k$, can be evaluated in terms of elements each of which has more than g i's in its upper and no less than h j's in its lower suffix.

Let $X^{i_1 \dots i_k}_{j_1 \dots j_k}$ be the fixed non-zero element of the k-th compound of X referred to in (7.4.17). Let us term desirable those elements of the k-th compound of X which satisfy any one of the three conditions of (7.4.21) and term undesirable all others. Clearly, every undesirable element falls into one of the three categories considered in the last several paragraphs above and consequently can be evaluated by means of (7.4.23), (7.4.25), or (7.4.27) according as it belongs to the first, second, or third category. We shall call such an evaluation of an undesirable element its appropriate evaluation. Now the following lemma is an immediate consequence of (7.4.24), (7.4.26), and (7.4.28).

(7.4.29) <u>Lemma.</u> Let $X^{r_1 \dots r_k}_{s_1 \dots s_k}$ be any undesirable element of the k-th compound of X; let g and h respectively denote the number of i's in its upper suffix and the number of j's in its lower suffix. Then, if, for $X^{r_1 \dots r_k}_{s_1 \dots s_k}$, $g = g_0$ and $h = h_0$, its appropriate evaluation expresses $X^{r_1 \dots r_k}_{s_1 \dots s_k}$ in terms of elements for which $g + h > g_0 + h_0$.

We need one more lemma.

(7.4.30) <u>Lemma.</u> For every element $X^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ of the k-th compound of X, $0 \leqq g + h \leqq 2k$. Furthermore, every element for which $2k - 2 \leqq g + h \leqq 2k$ is a desirable element.

The first part of (7.4.30) is obvious. The second part follows from the fact that the condition therein implies $X^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ satisfies one of the following three conditions: its upper suffix contains k i's, $g = k$; its lower suffix contains k j's, $h = k$; or its upper and lower suffixes respectively contain $k-1$ i's and $k-1$ j's, $g = h = k-1$.

Let $X^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ be an arbitrary, but fixed, undesirable element of the k-th compound of X, g_0 be the number of integers its upper suffix has in common with that of $X^{i_1 \cdots i_k}_{j_1 \cdots j_k}$, and h_0 be the number of integers their lower suffixes have in common. Then for $X^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ we have $g + h = g_0 + h_0$. Therefore, by (7.4.29), the appropriate evaluation, which we shall denote by E_1, of $X^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ contains only elements of the k-th compound of X for which $g + h \geqq g_0 + h_0 + 1$. Hence it is true, in particular, that $h + g \geqq g_0 + h_0 + 1$ for every undesirable element in E_1. Consequently, again by (7.4.29), if we replace each undesirable element in E_1 by its appropriate evaluation, we introduce only elements of the k-th compound of X for which $g + h \geqq g_0 + h_0 + 2$, and thus obtain a new evaluation, which we denote by E_2. Clearly, for every undesirable element in E_2, we have $g + h \geqq g_0 + h_0 + 2$. Let us continue the procedure of replacing the undesirable elements in each new evaluation by their respective appropriate evaluations to obtain the succeeding evaluation until we arrive at an evaluation, which we shall call $E(X_{s_1 \ldots s_k}^{r_1 \ldots r_k})$, containing no undesirable elements. This is possible since, at worst, we will have $E(X_{s_1 \ldots s_k}^{r_1 \ldots r_k}) = E_{(2k-g_0-h_0-2)}$. For, according to the outlined procedure, $E_{(2k-g_0-h_0-3)}$ can contain only undesirable elements for which

$$g + h \geqq g_0 + h_0 + (2k-g_0-h_0-3) = 2k-3.$$

Consequently, by (7.4.29), if we replace each undesirable element in it by that element's appropriate evaluation, we introduce only elements of the k-th compound of X for which $g + h = 2k-2$, in other words, by (7.4.30), we replace each of the undesirable elements by desirable elements. It follows from the very way in which it was obtained, that the right member of the equation

$$(7.4.31) \qquad X_{s_1 \ldots s_k}^{r_1 \ldots r_k} = E(X_{s_1 \ldots s_k}^{r_1 \ldots r_k})$$

is a well-defined, finite-valued function of desirable elements of the k-th compound of X.

Since the only properties of the X's used in deriving (7.4.31) are also, by hypothesis, properties of the B's, namely, their skew-symmetry and those implied by (7.4.1.3), and since, by (7.4.17), $B^i_j \neq 0$, we may replace the X's by B's if we replace the words "the k-th compound of X", wherever they occur, by the words "the matrix B" in the argument leading up to (7.4.31) to derive

$$B^{r_1 \cdots r_k}_{s_1 \cdots s_k} = E(B^{r_1 \cdots r_k}_{s_1 \cdots s_k}) , \tag{7.4.32}$$

where E is the same function as before, which evaluates each undesirable element of the matrix B in terms desirable elements.

Now let $X^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ be any element of the k-th compound of X and $B^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ be the corresponding element of the matrix B. If $X^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ is a desirable element then so is $B^{r_1 \cdots r_k}_{s_1 \cdots s_k}$, since they have the same suffixes, and, by (7.4.21),

$$X^{r_1 \cdots r_k}_{s_1 \cdots s_k} = B^{r_1 \cdots r_k}_{s_1 \cdots s_k} ;$$

otherwise, both are undesirable elements and we have

$$\begin{aligned} X^{r_1 \cdots r_k}_{s_1 \cdots s_k} &= E(X^{r_1 \cdots r_k}_{s_1 \cdots s_k}) && \text{by } (7.4.31) \\ &= E(B^{r_1 \cdots r_k}_{s_1 \cdots s_k}) && \text{by } (7.4.21) \\ &= B^{r_1 \cdots r_k}_{s_1 \cdots s_k} && \text{by } (7.4.32) \ . \end{aligned}$$

Therefore the k-th compound of X and the matrix B are equal element-wise, i.e., $X^{(k)} = B$. This completes the proof of the sufficiency of the set of conditions of (7.4.1).

Applying (3) of (1.4.11) to (7.4.1) we obtain the following theorem.

(7.4.33) <u>Theorem.</u> If B be a matrix of rank greater than k which satisfies the conditions of (7.4.1), then the matrix equation

$$X^{(k)} = B$$

is always solvable, there being precisely k solutions,

$$X = \omega_v X_0 \quad , \qquad v = 1,\ldots,k,$$

where the numbers $\omega_1,\ldots,\omega_k$ are the k k-th roots of unity and X_0 is determined by (7.4.10).

In the proof of (7.4.1) we needed algebraic closure only to insure the existence of a number a such that

$$a^k |A|^{k-1} = 1 \quad ,$$

where A was some non-singular, "distinguished" submatrix of B of order $k+1$. It follows therefore that the next theorem is true.

(7.4.34) <u>Theorem.</u> When B is a real matrix of rank greater than k which satisfies the conditions of (7.4.1), the matrix equation

$$X^{(k)} = B$$

has no real solution, one real solution, or two real solutions--one being the negative of the other--according as k is an even integer and $|A| < 0$ for every non-singular, "distinguished" submatrix A of B, k is an odd integer, or k is an even integer and $|A| > 0$ for some "distinguished" submatrix A of B. A has order $k+1$.

7.5. Secondary-diagonal Matrices. A matrix with all elements zero except some or all of those in the principal diagonal is called a diagonal matrix, see [1, p. 13]. Let us call a matrix with all elements zero except some or all of those in the secondary diagonal at right angles to the principal diagonal a secondary-diagonal matrix. Clearly, diagonal and secondary-diagonal matrices are square and are respectively defined by $a^i_j = 0$, $i \neq j$ and $a^i_j = 0$, $i + j \neq n + 1$, where $i,j = 1,\ldots,n$.

Let A and B be respectively diagonal and secondary-diagonal matrices of order n. We have for the i,j-th element $[A\ B]^i_j$ of $A\ B$

$$\begin{aligned}[A\ B]^i_j &= \sum_{v=1}^{n} a^i_v\ b^v_j \\ &= a^i_i\ b^i_j \quad \text{since } a^i_v = 0 \text{ for } v \neq i \\ &= 0 \text{ for } i + j \neq n + 1\end{aligned}$$

since $b^i_j = 0$ for $i + j \neq n + 1$. Hence $A\ B$ is a secondary-diagonal matrix of order n. We have also

$$[B\ A]^i_j = \sum_{\nu=1}^{n} b^i_v\, a^v_j$$

$$= b^i_j\, a^j_j$$

$$= 0 \qquad \text{for} \quad i + j \neq n + 1,$$

which shows that $B\ A$ is a secondary-diagonal matrix. Now let C be another secondary-diagonal matrix. We have

$$[B\ C]^i_j = \sum_{\nu=1}^{n} b^i_v\, c^v_j$$

$$= b^i_{n+1-i}\, c^{n+1-i}_j \quad \text{or} \quad b^i_{n+1-j}\, c^{n+1-j}_j$$

$$= 0 \qquad \text{for} \quad n + 1 - i + j \neq n + 1$$

or $n + 1 - j + i \neq n + 1$,

that is,

$$[B\ C]^i_j = 0 \qquad \text{for} \quad i \neq j\ .$$

Hence $B\ C$ is a diagonal matrix.

It is just as easy to show that the product of two diagonal matrices is again a diagonal matrix. Hence the following theorem is true.

(7.5.1) Theorem. The product of a diagonal and a secondary-diagonal matrix is a secondary-diagonal matrix; the product of two diagonal, or of two secondary-diagonal, matrices is a diagonal matrix.

Obviously the transpose of a secondary-diagonal matrix is a secondary-diagonal matrix. The same is true for the inverse of a non-singular secondary-diagonal matrix. For let A be the inverse of the secondary-diagonal matrix B or order n. Then we have

$$\sum_{\nu=1}^{n} a^i_\nu \, b^\nu_j = \sum_{\nu=1}^{n} b^i_\nu \, a^\nu_j = \delta^i_j \quad .$$

This reduces to

$$a^i_{n+1-j} \, b^{n+1-j}_j = b^i_{n+1-i} \, a^{n+1-i}_j = \delta^i_j \quad .$$

Hence $a^i_{n+1-j} = a^{n+1-i}_j = 0$ unless $i = j$ since $b^{n+1-j}_j \neq 0$ for $j = 1,\ldots,n$ by hypothesis. Therefore $a^u_v = 0$ unless $u+v = n+1$. That is, A is a secondary-diagonal matrix. Therefore the next theorem is true.

(7.5.2) Theorem. The transpose of a secondary-diagonal matrix and the inverse of a non-singular secondary-diagonal matrix are both secondary-diagonal matrices.

We need the following lemma in order to simplify the proof of the next theorem.

(7.5.3) Lemma. There is no non-singular matrix of order four whose second compound is a secondary-diagonal matrix.

The proof is by contradiction. We suppose the contrary and let $X = \left[x^i_j\right]$, $i,j = 1,2,3,4$, be such a matrix. Then

$$X^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & X^{12}_{34} \\ 0 & 0 & 0 & 0 & X^{13}_{24} & 0 \\ 0 & 0 & 0 & X^{14}_{23} & 0 & 0 \\ 0 & 0 & X^{23}_{14} & 0 & 0 & 0 \\ 0 & X^{24}_{13} & 0 & 0 & 0 & 0 \\ X^{34}_{12} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$X^{i_1 i_2}_{j_1 j_2} = \begin{vmatrix} x^{i_1}_{j_1} & x^{i_1}_{j_2} \\ x^{i_2}_{j_1} & x^{i_2}_{j_2} \end{vmatrix};$$

that is, $X^{i_1 i_2}_{j_1 j_2} = 0$ unless $(i_1 i_2 j_1 j_2)$ is some permutation of (1234) and, since $\|X\| \neq 0$, it follows from (1.4.4) that $X^{i_1 i_2}_{i_3 i_4} \neq 0$ for all permutations $(i_1 i_2 i_3 i_4)$ of (1234). Hence, by (7.4.1), see (7.4.1.4), we know that $X^{(2)}$ must contain at least one non-singular, "distinguished" sub-matrix of order three; let it be

$$A = \begin{bmatrix} X^{a_1 a_2}_{b_1 b_2} & X^{a_1 a_2}_{b_1 b_3} & X^{a_1 a_2}_{b_2 b_3} \\ X^{a_1 a_3}_{b_1 b_2} & X^{a_1 a_3}_{b_1 b_3} & X^{a_1 a_3}_{b_2 b_3} \\ X^{a_2 a_3}_{b_1 b_2} & X^{a_2 a_3}_{b_1 b_3} & X^{a_2 a_3}_{b_2 b_3} \end{bmatrix}.$$

Clearly, every non-singular submatrix of $X^{(2)}$ is also a secondary-diagonal matrix whose secondary diagonal is contained in that of $X^{(2)}$. Hence

$$A = \begin{bmatrix} 0 & 0 & X^{a_1a_2}_{b_2b_3} \\ 0 & X^{a_1a_3}_{b_1b_3} & 0 \\ X^{a_2a_3}_{b_1b_2} & 0 & 0 \end{bmatrix},$$

where, in particular, each of the 4-tuples $(a_1a_2b_2b_3)$, $(a_1a_3b_1b_3)$, and $(a_2a_3b_1b_2)$ is a permutation of (1234). From this we shall obtain the desired contradiction. Since $(a_1a_2b_2b_3)$ is a permutation of (1234), we deduce

$$a_1 \neq a_2, b_2, b_3.$$

Since $(a_1a_3b_1b_3)$ is a permutation of (1234), we deduce

$$a_1 \neq a_3, b_1, b_3 .$$

It follows therefore that

$$a_1 \neq a_2, a_3, b_1, b_2 ,$$

and hence, since $(a_2a_3b_1b_2)$ is a permutation of (1234), we have

$$a_1 \neq 1,2,3,4 .$$

But, since $(a_1a_2b_2b_3)$ is a permutation of (1234), a_1 is one of the integers $1,2,3,4$ which contradicts the preceding conclusion. This completes the proof of (7.5.3).

We are now in a position to prove the following theorem.

(7.5.4) <u>Theorem.</u> There is no matrix of rank greater than $k+1$, $k > 1$, whose k-th compound is a secondary-diagonal matrix.

Again the proof is by contradiction. We suppose the contrary and let X be a matrix of rank greater than $k+1$, $k > 1$, whose k-th compound is a secondary-diagonal matrix. Then, since X has rank greater than $k+1$, there must be a non-singular submatrix of X, let it be Y, of order $k+2$, whose k-th compound $Y^{(k)}$ is, by (1.4.4), a non-singular submatrix of $X^{(k)}$. Since the non-singular submatrices of $X^{(k)}$ are obviously secondary-diagonal matrices, it follows that $Y^{(k)}$ is a secondary-diagonal matrix. Since Y is of order $k+2$ and $|Y| \neq 0$, we have, by (1.9.5),

$$Y^{(2)} = |Y| G_{(k+2,2)} \left((Y^{(k)})^{-1} \right)' G'_{(k+2,2)}$$

Since $Y^{(k)}$ is a secondary-diagonal matrix, it follows from (7.5.2) that $\left((Y^{(k)})^{-1} \right)'$ is also a secondary-diagonal matrix. Moreover, it follows from (1.7.5) that $G_{(k+2,2)}$ is a secondary-diagonal matrix and consequently, by (7.5.2), so is $G'_{(k+2,2)}$. Then it follows from (7.5.1) that the product

$$G_{(k+2,2)}\left(\left(Y^{(k)}\right)^{-1}\right)' \; G'_{(k+2,2)}$$

is a secondary-diagonal matrix. Now it is obvious that the result of multiplying a secondary-diagonal matrix by a number is a secondary-diagonal matrix. Therefore it must be true that the second compound of Y is also a secondary-diagonal matrix.

Since Y is a non-singular matrix of order $k+2$, $k > 1$, there must be a non-singular submatrix of Y, let it be Z, of order 4, whose second compound $Z^{(2)}$ is, by (1.4.4), a non-singular submatrix of $Y^{(2)}$. But $Y^{(2)}$ is a secondary-diagonal matrix, and therefore, since the non-singular submatrices of a secondary-diagonal matrix are themselves secondary-diagonal matrices, so is $Z^{(2)}$ a secondary-diagonal matrix. Thus we have shown that the existence of a non-singular matrix X of rank greater than $k+1$, $k > 1$, whose k-th compound is a secondary-diagonal matrix implies the existence of a non-singular matrix Z of order four whose second compound is a secondary-diagonal matrix, contrary to (7.5.3). Hence no such matrix X exists. This completes the proof of (7.5.4).

The following theorem is an immediate consequence of (7.5.4) in view of (1.4.4).

(7.5.5) <u>Theorem.</u> No secondary-diagonal matrix of rank greater than $k+1$, $k > 1$, is a k-th compound.

It is easy to deduce the following corollary from (7.5.5).

(7.5.6) <u>Corollary.</u> Let $H_{n,k}$ be the secondary-diagonal matrix of order $C(n,k)$ with each secondary-diagonal element being unity and let $G_{n,k}$ be the secondary-diagonal matrix defined by (1.7.5). Then, if c be any number different from zero and $1 < k < n$, neither of the matrices $c\,H_{n,k}$ or $c\,G_{n,k}$ is a k-th compound.

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